Asymptotics of logarithms of distributions and means of integrals of powers of Brownian motion

© K. Bruce Erickson
Dept. of Mathematics, Box 354350
Seattle, WA 98195–4350

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1 Results

Let \{X(t)\} be standard linear Brownian motion with \(X(0) = 0\). This note contains a derivation of the principle part of the asymptotic limits of the logarithms of

\[ P\left[ \int_0^1 |X(s)|^p \, ds > a \right] \quad \text{and} \quad E \exp \left\{ \int_0^t |X(s)|^p \, ds \right\} \]

for \(a, t \to \infty\). Motivation for this work comes from the interpretation of the second quantity as the expected population size at time \(t\) in a branching process of Brownian particles each of which has a path dependent branching rate at place \(x\) equal to \(|x|^p\), \(-\infty < x < \infty\). See [6], pp. 201–206. However, this connection will not be discussed here. The first limit is required for the second.

**Theorem 1.** For any \(0 < p < \infty\), we have

\[
\lim_{a \to \infty} \log P \left[ \frac{\int_0^1 |X(s)|^p \, ds > a}{a^{2/p}} \right] = -\kappa(p),
\]

where

\[
\kappa(p) = \frac{1}{2} \min \left\{ \int_0^1 u'(x)^2 \, dx : u \in AC[0,1], u(0) = 0, \int_0^1 |u(x)|^p \, dx = 1 \right\}
\]

\[
= \frac{1}{4p} (1 + \frac{p}{2}) \frac{2-p}{p} B\left(\frac{1}{2}, \frac{1}{p}\right)^2, \quad B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt.
\]

**Theorem 2.** For \(0 < p < 2\),

\[
\lim_{t \to \infty} \log E \exp \left\{ \frac{\int_0^t |X(s)|^p \, ds}{t^{(2+p)/(2-p)}} \right\} = \kappa^*(p)
\]
where
\[
\kappa^*(p) = \left(1 - \frac{p}{2}\right) \left(\frac{p}{2\kappa(p)}\right)^{\frac{2}{p}} = \frac{2 - p}{2 + p} \left[\frac{2p^2}{B(1/2, 1/p)^2}\right]^{\frac{p}{2}}
\]

**Remarks.**

(i) If \(p > 2\) and \(t > 0\), or if \(p = 2\) but \(t \geq \pi \sqrt{2}/\sqrt{4}\), then \(E \exp\{\int_0^t |X(s)|^p \, ds\} \equiv \infty\) which follows immediately from [6], pp. 201–206. (In the branching process mentioned earlier, if \(p > 2\), then with probability 1, the number of particles becomes infinite in a finite time.)

(ii) A change of variables in the integral on the left in (1.1) and application of the scaling property of Brownian motion shows that for any fixed \(T > 0\),

\[
\lim_{b \to \infty} \frac{\log P \left[ \int_0^T |X(s)|^p \, ds > b \right]}{b^{2/p}} = -\frac{\kappa(p)}{T^{1+2/p}}
\]

(iii) Theorem 1 is stated in the form useful for proving (1.4). Using the functional \(\|X_{[0,1]}\|_p = \left(\int_0^1 |X_s|^p \, ds\right)^{1/p}\) transforms the limit (1.1) into the more pleasing:

\[
\lim_{b \to \infty} \frac{\log P \left[ \|X_{[0,1]}\|_p > b \right]}{b^{2/p}} = -\kappa(p), \quad 0 < p < \infty.
\]  

In this form (1.5) is seen to be correct for \(p = \infty\) (sup norm) yielding the well known:

\[
\lim_{b \to \infty} \frac{\log P \left[ \max_{0 \leq t \leq 1} |X(t)| > b \right]}{b^2} = -\frac{1}{2}.
\]

See [5]. Note that \(\lim_{p \to \infty} \kappa(p) = 1/2\) and that a minimizer for the variational problem:

\[
\min \left\{ \frac{1}{2} \int_0^1 |u'|^2 \, dx, \, u \in AC[0,1], \, u(0) = 0, \, \|u\|_\infty = 1 \right\}
\]

is easily seen to be \(u(x) = x\), so that (1.2), with this modification, also holds.

The limit (1.5) also partially extends to \(p = 0\) where

\[
\|u\|_0 = \begin{cases} 
\exp \left( \int_0^1 \log |u(x)| \, dx \right) & \text{if } u \neq 0 \text{ a.e. (Lebesgue)} \\
0 & \text{otherwise}
\end{cases} \quad \text{is} \quad \lim_{p \to 0^+} \|u\|_p, \quad u \in C[0,1].
\]

For bounded \(u\) the \(\int \log |u| \, dx\) makes sense but may equal \(-\infty\) in which case also \(\|u\|_0 = 0\). The limit exists because \(p \to \|u\|_p\) is an increasing function and this monotonicity together with (1.5) and (1.3) shows that

\[
\limsup_{a \to \infty} \frac{\log P \left[ \int_0^1 \log |X_t| \, dt > a \right]}{e^{2a}} = \limsup_{b \to \infty} \frac{\log P \left[ \|X_{[0,1]}\|_0 > b \right]}{b^2} \leq -\lim_{p \to 0^+} \kappa(p) = -\frac{\pi e}{4}.
\]

No doubt a limit equality holds here as in (1.5), but I have no proof. The difficulty is that \(\|\cdot\|_0\) is discontinuous in the uniform topology of \(\{u \in C[0,1], u(0) = 0\}\) and the obvious large deviations principle does not give a useful lower bound. See Step 1 in the proof of (1.1). The variational formula (1.2), where \(\kappa(0) = \pi e/4\) and the integral \(\int |u|^p\) replaced by \(\|\cdot\|_0\), is also valid but due to its length and marginal interest I omit the proof.)
2 Proof of Theorem 1

Theorem 1 is a consequence of a version of Schilder’s theorem on large deviations.

Let $C_0$ denote the Banach space of continuous functions on $[0, 1]$ vanishing at 0 equipped with the supremum norm $\| \cdot \|$, and let $\mathcal{H}$ denote those functions in $C_0$ which are absolutely continuous with a square integrable derivative. For $u \in C_0$, we write

$$J(u) = \begin{cases} \frac{1}{2} \int_0^1 |u'(x)|^2 \, dx & u \in \mathcal{H} \\ \infty & \text{otherwise} \end{cases}$$

**Step 1** Denote by $X_{[0,1]}$ the random variable with values in $C_0$ given by the Brownian path segment $\{X_t; 0 \leq t \leq 1\}$. The distribution of $X_{[0,1]}$ induced by $P$ is simply Wiener measure on $C_0$. Let $D$ denote a Borel set of functions in $C_0$. Let $D^0$ denote the interior of $D$ as determined by the topology of $C_0$ and $\overline{D}$ its closure; then

$$-\inf\{ J(u) : u \in \mathcal{H} \cap D^0 \} \leq \liminf_{\varepsilon \to 0^+} \varepsilon \log P\left[ \sqrt{\varepsilon} X_{[0,1]} \in D \right] \leq \limsup_{\varepsilon \to 0^+} \varepsilon \log P\left[ \sqrt{\varepsilon} X_{[0,1]} \in D \right] \leq -\inf\{ J(u) : u \in \mathcal{H} \cap \overline{D} \} \tag{2.1}$$

This, of course, is a version of Schilder’s Large Deviations Principle. A proof may be found in many places: See, for example, [4] or [3].

**Step 2** Let $D = \{ u \in C_0 : \int_0^1 |u|^p > 1 \}$. The set $D$ is open in the topology of $C_0$ and its closure is the set $\overline{D} = \{ u \in C_0 : \int_0^1 |u|^p \, dx \geq 1 \}$.

Since $C_0$ is a metric space, it suffices, for the first assertion, to show that $D^c = \{ u \in C_0 : \int_0^1 |u|^p \leq 1 \}$ is sequentially closed. Let $\{u_n\}$ be a sequence in $D^c$ which converges uniformly to a $u \in C_0$. Then $\int_0^1 |u|^p = \lim_{n \to \infty} |u_n|^p \leq 1$ by Dominated Convergence. The second assertion is proved similarly.

**Step 3** Write $a = \varepsilon^{-p/2}$, then

$$P\left[ \sqrt{\varepsilon} X_{[0,1]} \in D \right] = P\left[ \int_0^1 \sqrt{\varepsilon} |X(s)|^p \, ds > 1 \right] = P\left[ \int_0^1 |X(s)|^p \, dt > a \right]$$

and the same equation holds if $D$ is replaced by $\overline{D}$ and $>$ by $\geq$. Thus (2.1) transforms to the equivalent:

$$-\inf\{ J(u) : u \in \mathcal{H} \cap D \} \leq \liminf_{a \to \infty} \frac{\log P\left[ \int_0^1 |X_s|^p \, ds > a \right]}{a^{2/p}} \leq \limsup_{a \to \infty} \frac{\log P\left[ \int_0^1 |X_s|^p \, ds > a \right]}{a^{2/p}} \leq -\inf\{ J(u) : u \in \mathcal{H} \cap \overline{D} \} \tag{2.2}$$

**Step 4** Proof of (1.2). Let $D^* = \overline{D} \setminus D = \{ u : \int |u|^p = 1 \}$; then

$$\kappa = \inf\{ J(u) : u \in \mathcal{H} \cap D \} = \inf\{ J(u) : u \in \mathcal{H} \cap D^* \} = \inf\{ J(u) : u \in \mathcal{H} \cap \overline{D} \}$$

3
Denote by $\kappa^*$ the second inf. Since $\overline{D} = D^* \cup D$, it suffices to show $\kappa^* = \kappa$. Note that both $\kappa$ and $\kappa^*$ must be finite because $u = 2(p+1)^{1/p}x$ is in $D$ and $z = (p+1)^{1/p}x$ is in $D^*$.

Let $\varepsilon > 0$ be fixed but arbitrary. On the one hand, there exists $u \in D \cap \mathcal{H}$ such that $J(u) \leq \kappa + \varepsilon$ and if $c = (\int |u|^p)^{1/p}$, then $c > 1$, $\varepsilon = u/c \in D^* \cap \mathcal{H}$, and

$$\kappa^* \leq J(z) = \frac{1}{2} \int z'^2 = c^{-2}J(u) < J(u) \leq \kappa + \varepsilon.$$  

The arbitrariness of $\varepsilon > 0$ implies $\kappa^* \leq \kappa$. On the other hand, choose $z \in D^* \cap \mathcal{H}$ with $J(z) \leq \kappa^* + \varepsilon$ and put $u = z(1 + \varepsilon)^{1/2}$; then $\int |u|^p = (1 + \varepsilon)^{p/2} > 1$, so $u \in D \cap \mathcal{H}$ and

$$\kappa \leq J(u) = \frac{1}{2} \int u'^2 = (1 + \varepsilon)J(z) \leq (1 + \varepsilon)(\kappa^* + \varepsilon).$$

Now the arbitrariness of $\varepsilon$ implies $\kappa \leq \kappa^*$. Therefore $\kappa = \kappa^*$ and (1.2) is proved.

**Step 5** The limit (1.1) holds: The result of Step 4 forces equality throughout (2.2).

**Step 6** Proof of (1.3). For a fixed $0 < p < \infty$, consider the following inequality:

$$\int_0^1 |y(x)|^p \, dx \leq K \left( \int_0^1 |y'(x)|^2 \, dx \right)^{p/2} \quad (2.3)$$

for $y$ absolutely continuous, $y(0) = 0$, and $K$ some number. According to a special case of a theorem of Boyd, page 376 of [1], the inequality (2.3) holds for all functions $y$ in the given class if and only $K \geq K(p, 0, 2)$ where (after some simplification):

$$K(p, 0, 2) = 2p^{p-1}(1 + 2/p)^{p/2-1}B(1/2, 1/p)^{-p}. \quad (B \text{ is the standard complete beta function as in (1.3).})$$

Moreover, equality actually holds in (2.3) for $K = K(p, 0, 2)$ for constant multiples of a single function $y_0$ (and only for such multiples) which is a solution to the problem

$$\lambda y'^2 + y^p = (1 + 2/p)\lambda, \quad y > 0 \text{ on } (0, 1], \quad y(0) = 0, \quad \int_0^1 |y'|^2 \, dx = 1,$$  

in which $\lambda$ equals the largest number for which this problem has a solution. (Boyd shows that $0 < \lambda < \infty$ (one of the tricky parts), and that $K(p, 0, 2) = 2\lambda/p$. See Theorem 1 and inequality (5) of [1], p.368.)

From (1.2) we have

$$\int_0^1 |u|^p \, dx = 1 \leq \frac{1}{2\kappa(p)} \int_0^1 u'^2 \, dx$$

for all admissible $u$ (i.e., those $u \in AC[0, 1], u(0) = 0, \int |u|^p = 1$). But we also know from (2.3) that

$$1 = \int_0^1 |u|^p \, dx \leq K(p, 0, 2) \left( \int_0^1 |u'|^2 \, dx \right)^{p/2}$$

for all admissible $u$ and that equality actually occurs when $u = Cy_0$ for some number $C$. But this says that the infimum of $\int_0^1 |u'|^2 \, dx$, $u$ admissible, must in fact equal $1/K(p, 0, 2)^{2/p}$ and therefore $\frac{1}{p} \inf = \kappa(p) = \frac{1}{2} K(p, 0, 2)^{-2/p}$ which evaluates to the expression in (1.3) as the reader may check.
Comment. The inequality can also be established using direct methods of the Calculus of Variations. Additional complications arise for $0 < p < 1$ because the functional $\int |u|^p$ no longer has a well behaved variation. However, these complications are not insurmountable and eventually one does get $\kappa(p)$ in all cases. Boyd’s method bypasses this problem and deduces the inequality more directly from a non-linear differential equation, equation (2.4) in this case.

3 Proof of Theorem 2

Step 1 Write $Y = \int_0^1 |X(s)|^p \, ds = \|X[0,1]\|^p_p$; then

$$E \exp \left( \int_0^t |X(s)|^p \, ds \right) = E \exp \left( t^{1+p/2} Y \right), \quad 0 < p < 2.$$  \hspace{1cm} (3.1)

The proof is immediate from the Brownian scaling property: For any number $c > 0$, the process $[\sqrt{c} X(t/c); \ t \geq 0; \ P]$ has the same probability law as that of the process $[X(t); \ t \geq 0; \ P]$. Therefore, the random variable $\int_0^1 |X(s)|^p \, ds$ has, with respect to $P$, exactly the same distribution as $t^{p/2} \int_0^t |X(s/t)|^p \, ds = t^{1+p/2} Y$ with respect to $P$. For the necessity of the restriction $0 < p < 2$, see Remark (i) following Theorem 2.

Step 2 Write $z = t^{1+p/2} > 0$. For any $\varepsilon > 0$, put $m_1 = \kappa(p) - \varepsilon$, $m_2 = \kappa(p) + \varepsilon$; then there is a number $a_\varepsilon > 0$, depending only on $\varepsilon$, such that for all $a \geq a_\varepsilon$

$$z \int_0^\infty \exp\{za - m_2 a^{2/p}\} \, da - e^{a_\varepsilon z} < E e^{zy} < z \int_0^\infty \exp\{za - m_1 a^{2/p}\} \, da + e^{a_\varepsilon z}.$$  \hspace{1cm} (3.2)

By Theorem 1 there exists $a_\varepsilon$ such that

$$E e^{zy} = 1 + z \int_0^\infty e^{za} P[Y > a] \, da < 1 + z \int_0^{a_\varepsilon} e^{za} P[Y > a] \, da + z \int_{a_\varepsilon}^\infty e^{za - m_1 a^{2/p}} \, da$$

$$\leq 1 + z \int_0^{a_\varepsilon} e^{za} \, da + z \int_0^\infty e^{za - m_1 a^{2/p}} \, da$$

and the righthand side of (3.2) follows. Similarly

$$E e^{zy} > z \int_{a_\varepsilon}^\infty e^{za - m_2 a^{2/p}} \, da \geq z \int_0^\infty e^{za - m_2 a^{2/p}} \, da - z \int_0^{a_\varepsilon} e^{za} \, da$$

$$= z \int_0^\infty e^{za - m_2 a^{2/p}} \, da + 1 - e^{a_\varepsilon z}$$

and the lefthand side of (3.2) follows.

Step 3 The limit (1.4).
On making the substitutions \(a = sx\), \(s = z^{p/(2-p)} = t^{\sigma p/2}\), \(\sigma = (2 + p)/(2 - p)\), (recall that \(z = t^{1+p/2}\)), in the integral of (3.2), we find
\[
z \int_0^\infty e^{z a - m x^{2/p}} \, da = t^{\sigma} \int_0^\infty e^{t^\sigma (x/m - x^{2/p})} \, dx, \quad m = m_1, m_2.
\]
The integral on the right has the basic form to which Laplace’s method of asymptotic evaluation applies. (See, for example, [2], Chapter 4.) Write \(v(x) = x - m x^{2/p}\), and \(c\) for the solution to \(dv(x)/dx = 0\); then
\[
\int_0^\infty e^{t^\sigma (x - m x^{2/p})} \, dx = R t^{-\sigma/2} e^{v(c)t^\sigma} [1 + \delta(t)], \quad R = \sqrt{(2\pi/v''(c))},
\]
for some \(\delta\), depending on \(t\), \(m\) and \(p\), but satisfying \(\lim_{t \to \infty} \delta(t) = 0\), i.e.:
\[
z \int_0^\infty e^{z a - m x^{2/p}} \, da = R t^{\sigma/2} e^{v(c)t^\sigma} [1 + \delta(t)], \quad t \to \infty, \quad z = t^{1+p/2}.
\]
The important constants in this expression are \(c = (p/2m)^{p/(2-p)}\) and
\[
v(c) = (p/2m)^{p/(2-p)} - m (p/2m)^{2/(2-p)} = (1 - p/2) (p/2m)^{p/(2-p)} > 0
\]
Put \(v_1 = v(c)\) with \(m_1 = \kappa + \varepsilon\) replacing \(m\) (ditto \(R_1\) and \(\delta_1\)). From (3.1) and (3.3) we get
\[
E e^{zY} < e^{\sigma z} + z \int_0^\infty e^{z a - m_1 x^{2/p}} \, da
\]
\[
= e^{\sigma z} + R_1 t^{\sigma/2} e^{v_1 t^\sigma} [1 + \delta_1(t)]
\]
\[
= R_1 t^{\sigma/2} e^{v_1 t^\sigma} [1 + \delta_1(t)] [1 + \delta_1'(t)], \quad \delta_1'(t) = \frac{e^{\sigma z - v_1 t^\sigma}}{R_1 t^{\sigma/2} [1 + \delta_1(t)]}.
\]
But \(1 + p/2 < (2 + p)/(2 - p) = p\), so \(z = t^{1+p/2} = o(t^{\sigma})\) and \(\delta_1'(t) \to 0\) as \(t \to \infty\), which leads to the conclusion:
\[
\lim_{t \to \infty} \sup \frac{\log E e^{[j]X(s)p \, ds}}{t^{\sigma}} \leq v_1 = (1 - p/2) \left( \frac{p}{2\kappa(p) - 2\varepsilon} \right)^{\frac{\sigma}{p-\sigma}}, \quad \sigma = \frac{2 + p}{2 - p}.
\]
A similar argument using (3.3) and the lefthand side of (3.2) shows that
\[
\lim_{t \to \infty} \inf \frac{\log E e^{[j]X(s)p \, ds}}{t^{\sigma}} \geq (1 - p/2) \left( \frac{p}{2\kappa(p) + 2\varepsilon} \right)^{\frac{\sigma}{p-\sigma}}.
\]
Making \(\varepsilon \downarrow 0\) in these bounds finishes the proof of (1.4).
References


