① Given $\sum_{\iota=1}^{\infty} a_\iota$ suppose there is $r \in \mathbb{R}$, $0 \le r < 1$ and $M \in \mathbb{N}$ such that $\forall \iota \ge M$ $|a_\iota|^{1/\iota} < r$

Prove $\sum_{\iota=1}^{\infty} a_\iota$ converges

Proof: by assumption $\forall \iota \ge M$ $|a_\iota| \le r^\iota$

therefore $\sum_{\iota=1}^{\infty} |a_\iota|$ converges by comparison with the geometric series $\sum_{\iota=1}^{\infty} r^\iota$, which converges since $0 \le r < 1$

Therefore $\sum_{\iota=1}^{\infty} a_\iota$ converges absolutely

Note that here we are assuming the more general version of the comparison test only requiring $\forall \iota \ge M$ $0 \le a_\iota \le b_\iota$ and not $\forall \iota \ge 1$ $0 \le a_\iota \le b_\iota$. See problem ② below for a proof that "the first M terms" of a series do not affect the convergence of a series.

② $\forall M \in \mathbb{N}$ $\sum_{\iota=1}^{\infty} a_\iota$ converges $\iff$ $\sum_{\iota=M}^{\infty} a_\iota$ converges

Proof: let $s_n = \sum_{\iota=1}^{n} a_\iota$, $t_n = \sum_{\iota=M}^{n} a_\iota$ (for $n > M$)

Then $s_n = \underbrace{a_1 + a_2 + \cdots + a_{M-1}}_{\text{this is just a constant } C} + t_n$

So we can write $s_n = C + t_n$ (for $n > M$)

then if $\sum_{i=1}^{\infty} a_i$ converges $\lim_{n\to+\infty} s_n = \alpha$ for some $\alpha \in \mathbb{R}$

so $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} (s_n - C) = \alpha - C \in \mathbb{R}$

so $\{t_n\}$ converges so $\sum_{i=M}^{\infty} a_i$ converges

viceverse if $\sum_{i=M}^{\infty} a_i$ converges then

$\lim_{n\to+\infty} t_n = \beta$ for some $\beta \in \mathbb{R}$ so $\lim_{n\to+\infty} s_n =$

$= \lim_{n\to+\infty} (t_n + C) = \beta + C$ so $\{s_n\}$ converges
so $\sum_{i=1}^{\infty} a_i$ converges

(Intuitively $\sum_{i=1}^{\infty} a_i = a_1 + \cdots + a_{M-1} + \sum_{i=M}^{\infty} a_i$, but we want to be careful in expleinig what it means when one of the series diverges, in this case we mean the other diverges too)