1) $\frac{3^k}{4^k+4} < \left(\frac{3}{4}\right)^k$ Therefore $\sum_{k=1}^{\infty} \frac{3^k}{4^k+4}$ converges by comparison with the geometric series $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k$

2) $$\lim_{k\to+\infty} \frac{\frac{3^k+1}{4^k-3}}{\left(\frac{3}{4}\right)^k} = \frac{3^k\left(1+\frac{1}{3^k}\right)}{4^k\left(1-\frac{3}{4^k}\right)}\cdot\frac{4^k}{3^k} = \frac{1+\left(\frac{1}{3}\right)^k}{1-3\left(\frac{1}{4}\right)^k} = 1 > 0$$

Therefore the series $\sum_{k=1}^{\infty}\frac{3^k+1}{4^k-3}$ (which is a positive series) converges by limit test with $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k$

3) Try ratio test: $$\lim_{k\to+\infty} \frac{\frac{3^{k+1}(k+1)}{4^{k+1}}}{\frac{3^k k}{4^k}} = \frac{3^{k+1}}{4^{k+1}}\cdot\frac{4^k}{3^k}\quad\frac{k+1}{k} =$$
$$= \frac{3}{4}\,\frac{\cancel{k}(1+1/k)}{\cancel{k}} = \frac{3}{4} < 1$$

Therefore the series converges (we could say absolutely but it is a non negative series anyways)

4) We could use the ratio test again or use comparison test: $\frac{3^k}{4^k\cdot k} < \frac{3^k}{4^k}$ therefore $\sum_{k=1}^{\infty}\frac{3^k}{4^k k}$ converges by comparison test with the geometric series $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k$

5) Try ratio test: $$\lim_{k\to+\infty}\frac{\frac{(k+1)!\,(k+2)!}{(3(k+1))!}}{\frac{k!\,(k+1)!}{(3k)!}} = \frac{(k+1)!\,(k+2)!\;(3k)!}{(3k+3)!\;k!\,(k+1)!} =$$

$$= \frac{(k+1)(k+2)}{(3k+1)(3k+2)(3k+3)} = \frac{k^2\,(1+1/k)(1+2/k)}{k^3\,(3+1/k)\left(3+\frac{2}{k}\right)\left(3+\frac{3}{k}\right)} = \frac{1}{k}\cdot\frac{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)}{\left(3+\frac{1}{k}\right)\left(3+\frac{2}{k}\right)\left(3+\frac{3}{k}\right)} = 0$$

so the series converges

6) The series converges absolutely by ratio test (see problem 3)

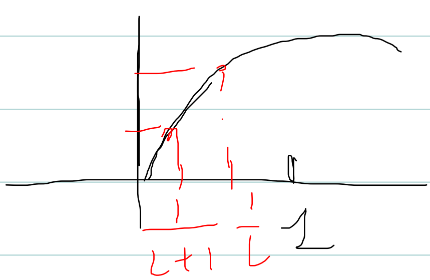
7) $\left|\frac{\sin(L)\, 3^L L}{4^L}\right| \leq \frac{3^L \cdot L}{4^L}$, by problem 3 $\sum_{L=1}^{\infty} \frac{3^L L}{4^L}$ converges, so by comparison test, $\sum_{L=1}^{\infty} \left|\frac{\sin(L) 3^L L}{4^L}\right|$ converges, so $\sum_{L=1}^{\infty} \frac{\sin(L)\, 3^L \cdot L}{4^L}$ converges absolutely.

8) $\lim_{L \to +\infty} \cos\left(\frac{1}{L}\right) = 1$ (we'll review this in ch 3 $\frac{1}{L} \to 0$, $\cos(0)=1$, and $\cos(x)$ is a continuous function; you do not have to worry about functions and continuity for the midterm.)

Therefore if $a_L = (-1)^L \cos\left(\frac{1}{L}\right)$ $a_{2L} \to 1$ $a_{2L+1} \to -1$ and $\lim_{L \to +\infty} a_L$ does not exists and in particular is not 0, so $\sum_{L=1}^{\infty} (-1)^L \cos\left(\frac{1}{L}\right)$ diverges

9) $\lim_{L \to +\infty} \sin\left(\frac{1}{L}\right) = 0$ (continuity of $\sin(x)$ function)

$\sin\left(\frac{1}{L+1}\right) \leq \sin\left(\frac{1}{L}\right)$ since $\frac{1}{L+1} < \frac{1}{L}$ and $\sin x$



is increasing on $[0,1]$ and $\frac{1}{L} \in [0,1]$

so by alternating series test $\sum_{L=1}^{\infty} (-1)^L \sin\left(\frac{1}{L}\right)$ is convergent

The series in not absolutely convergent since

$$\sum_{i=1}^{\infty} \left|(-1)^i \sin\left(\frac{1}{i}\right)\right| = \sum_{i=1}^{\infty} \sin\left(\frac{1}{i}\right) \qquad \left(\text{since } 0 < \frac{1}{i} \le 1 \quad \sin\frac{1}{i} \ge 0\right)$$

and $\lim_{i\to+\infty} \dfrac{\sin\left(\frac{1}{i}\right)}{\frac{1}{i}} = 1$ so by limit

test since $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges, then $\sum_{i=1}^{\infty} \sin\left(\frac{1}{i}\right)$

diverges also

10) $\dfrac{1}{\sqrt{i^2+1}} = \dfrac{1}{i\sqrt{1+\frac{1}{i^2}}} \to 0$

$\dfrac{1}{\sqrt{(i+1)^2+1}} \le \dfrac{1}{\sqrt{i^2+1}}$

Therefore $\sum_{i=1}^{\infty} (-1)^i \frac{1}{\sqrt{i^2+1}}$ converges by the alternating series test. Does $\sum_{i=1}^{\infty} \left|(-1)^i \frac{1}{\sqrt{i^2+1}}\right| = \sum_{i=1}^{\infty} \frac{1}{\sqrt{i^2+1}}$ converge? No

by the limit test, since $\lim_{i\to+\infty} \dfrac{\frac{1}{\sqrt{i^2+1}}}{\frac{1}{i}} = \dfrac{i}{i\sqrt{1+1/i^2}} \to 1$

so $\sum_{i=1}^{\infty} (-1)^i \frac{1}{\sqrt{i^2+1}}$ is convergent, but not absolutely convergent

11) This is the same as 10) since $(-1)^{3i}$ is $\begin{cases} -1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$

so $(-1)^{3i} = (-1)^i$

Since $0 \le b_i \le a_i$ Then by the comparison test $\sum_{i=1}^{\infty} b_i$ converges

This would not be necessarily true without assuming $a_i \geq 0$

For example $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ converges

by alternating series test

but $\sum_{i=1}^{\infty} \frac{1}{2i-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$

$$\geq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2i} = \sum_{i=1}^{\infty} \frac{1}{2} \cdot \frac{1}{i}$$

which diverges