1. (8 points)

(a) Complete the definition.

i. A sequence $\{a_n\}$ *converges* to a real number a if ...

$\forall \varepsilon > 0 \quad \exists M \in \mathbb{N} \quad \forall n \geq M \quad |a_n - a| < \varepsilon$

ii. A sequence $\{a_n\}$ is *Cauchy* if...

$\forall \varepsilon > 0 \quad \exists M \in \mathbb{N} \quad \forall m, n \geq M \quad |a_m - a_n| < \varepsilon$

(b) State the Cauchy Criterion for Convergence.

We did not name it, it is the th that says a sequence converges iff it is Cauchy

(c) Give an example of a sequence that is not Cauchy. Justify your answer.

Any divergent sequence

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2. (4 points each)

(a) FALSE

Counter-example: The sequence $\left\{1 - \frac{1}{n}\right\}$ is strictly increasing since, for all natural numbers n,

$$0 < n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow -\frac{1}{n} < -\frac{1}{n+1} \Rightarrow 1 - \frac{1}{n} < 1 - \frac{1}{n+1}.$$

By the limit properties,

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1.$$

So, $\left\{1 - \frac{1}{n}\right\}$ converges and therefore has a convergent subsequence (itself). (Alternatively, $\left\{1 - \frac{1}{n}\right\}$ converges and is therefore bounded and has a convergent subsequence by the Bolzano-Weierstrass Theorem.)

(b) TRUE

Proof: Suppose $\{a_n\}$ is an increasing sequence of negative terms and $\{b_n\}$ is a decreasing sequence of non-negative terms. Then, for each $n \in \mathbb{N}$, $a_n \le a_{n+1} < 0$, which implies $0 < -a_{n+1} \le -a_n$, and $0 \le b_{n+1} \le b_n$. So, for each $n \in \mathbb{N}$, $0 \le -a_{n+1}b_{n+1} \le -a_n b_n$, which implies $a_n b_n \le a_{n+1}b_{n+1} \le 0$. Thus, $\{a_n b_n\}$ is increasing. □

(c) TRUE

Proof: Suppose $n \in \mathbb{N}$ and a and b are real numbers such that $a \ge b \ge 0$. Then for each integer k such that $0 \le k \le n-1$, $a^{n-1-k} \ge b^{n-1-k}$. We then have, by the Difference of Powers formula:

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k \ge (a-b)\sum_{k=0}^{n-1} b^{n-1} = (a-b)nb^{n-1}.$$

□

(d) TRUE

Proof: Suppose $x, y \in \mathbb{R}$ and $x < y$. By the Rational Density Theorem, there is a rational number q such that $x < q < y$. Again by the Rational Density Theorem, there are rational numbers q_1 and q_2 such that $x < q_1 < q < q_2 < y$. Therefore, there are at lease three rational numbers in the interval (x, y). □

3. (a) (2 points)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{4n+1} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{4 + \frac{1}{n}} = \frac{1}{4}$$ (by the Quotient and Sum Properties).

(b) (4 points)

Proof: Choose an arbitrary $\epsilon > 0$. For every $n \in \mathbb{N}$,

$$\left|a_n - \frac{1}{4}\right| = \left|\frac{n-1}{4n+1} - \frac{1}{4}\right| = \frac{5}{16n+4} < \frac{5}{16n} < \frac{1}{n}.$$

By the Archimedean Property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Suppose $n \geq N$. Then

$$\left|a_n - \frac{1}{4}\right| < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

□

(c) (4 points) **Claim:** $\{a_n\}$ is monotonically increasing.

Proof: For each $n \in \mathbb{N}$,

$$a_{n+1} - a_n = \frac{n}{4n+5} - \frac{n-1}{4n+1} = \frac{5}{(4n+5)(4n+1)} \geq 0.$$

So, $a_{n+1} \geq a_n$ and $\{a_n\}$ is increasing. □

(d) i. (6 points) **Claim:** $\inf(S) = 0$ and $\sup(S) = \frac{1}{4}$.

Proof: For each $n \in \mathbb{N}$, $a_n \geq 0$ since $n - 1 \geq 0$ and $4n + 1 > 0$. So, 0 is a lower bound for S. Since $0 = a_1$, if ℓ is a lower bound for S, then $\ell \leq 0$. Thus $0 = \inf(S)$. Since $\{a_n\}$ converges, it is bounded. Since it's also monotone, by the Monotone Convergence Theorem, $\sup(S) = \lim_{n\to\infty} a_n = \dfrac{1}{4}$. □

ii. (2 points) No. Suppose there is an $n \in \mathbb{N}$ such that $\frac{n-1}{4n+1} = \frac{1}{4}$. Then $4n-4 = 4n+1$, which implies $-4 = 1$, a contradiction. Thus, $\sup(S) = \frac{1}{4} \notin S$ and S has no maximum.

(e) **Proof:** Suppose $\{a_n\}$ converges to a. There is an $N \in \mathbb{N}$ such that, if $n \geq N$, then $|a_n - a| < 1$. For $n \geq N$,

$$|a_n| = |a_n - a + a| \leq |a_n + a| + |a| < 1 + |a|.$$

Let $M = \max\{1 + |a|, |a_1|, |a_2|, ..., |a_{N-1}\}$. Then for all $n \in \mathbb{N}$, $|a_n| \leq M$ and thus $\{a_n\}$ is bounded. □