FINAL

Math 327A

name

You must show all work for full credit. Use the backs of the test pages as necessary.

you should be able to prove $\{x_n\}$ converges. you do not need to calculate what it converges to

1. Find the limit of the sequence $\{x_n\}$ defined inductively by $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2}^{x_n}$. Justify your answer. You may assume that $x_n < \sqrt{2}^{x_n}$ for any term x_n of this sequence.

since $x_n < \sqrt{2}^{x_n} = x_{n+1}$ x_n is increasing, $\{x_n\}$ is also bounded since

$\forall n \in N$ $x_n \le 2$ Proof by induction

1) For $n=1$ $\sqrt{2} \le 2$

2) Assume $x_n \le 2$ then $x_{n+1} = \sqrt{2}^{x_n} \le \sqrt{2}^2 = 2$

So $\{x_n\}$ is convergent to some limit ℓ. we must have

$\ell = \sqrt{2}^{\ell}$. The function $f(x) = x - \sqrt{2}^x$ is continuous and increasing (check f')

between $\sqrt{2}$ and 2 $f(2) = 0$ and 2 must then be the only 0 so $\ell = 2$

2. Use the *definition* of limit to compute the limit of $n/(n^2+1)$ as $n \to \infty$. Justify your answer.

Guess $\lim_{n \to \infty} \frac{n}{n^2+1} = 0$ proof

Given $\varepsilon > 0$ take $M > \frac{1}{\varepsilon}$ then if $n \ge M$ $\left|\frac{n}{n^2+1}\right| = \frac{n}{n^2(1+\frac{1}{n^2})} =$

$= \frac{1}{n} \cdot \frac{1}{1+\frac{1}{n^2}} < \frac{1}{M} < \varepsilon$

3. Let $\{x_n\}$ be a sequence of real numbers converging to $\pi/2$. Show that $\{\cos x_n\}$ converges to 0.

We proceed in the hw that $\sin(x)$ is a continuous function

since $\cos x = \sin(\frac{\pi}{2} - x)$, $\cos x$ is continuous too so

if $x_n \to \frac{\pi}{2}$ $\cos x_n \to \cos\frac{\pi}{2} = 0$

4. The following are *incorrect* statements of theorems discussed in class. In each case give the *correct* statement of the theorem.

a. If $f_n(x) \to f(x)$ on $[a, b]$ and f_n, f are continuous, then the convergence is uniform.

b. If $\{f_n(x)\}$ converges uniformly to the function $f(x)$ on $[a, \infty)$ and the f_n are continuous, then $\{\int_a^\infty f_n(x)dx\}$ converges to $\int_a^\infty f(x)dx$.

c. If $f_n(x) \le a_n$ on $[a, b]$ and a_n converges to 0, then $f_n(x)$ converges to 0 uniformly.

skip

5. Write down a power series which converges exactly for $|x| < 2$ and compute its sum explicitly.

skip

$\left(\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \right)$

6. Work out a power series expansion of $\tan^{-1}(x^2)$ valid for $|x| < 1$.

Skip

7. Decide whether the following sequences $f_n(x)$ of functions converge uniformly to the given function $f(x)$ on the given interval.

a. $f_n(x) = n/(x+n)$, $f(x) = 1$, on $[1, \infty)$. NO

b. $f_n(x) = (\tan^{-1} nx)/n^2$, $f(x) = 0$, on $[0, \infty)$. YES

SKip c. $f_n(x) = \sum_{i=0}^{n} x^i$, $f(x) = 1/(1-x)$, on $[-3/4, 1/2]$ es Y -)

a) Take $\varepsilon = \frac{1}{2}$, then given M take $n > M$ and $x > n$ so $\left|\frac{n}{x+n} - 1\right| = \frac{x}{x+n} > \frac{1}{2}$

b) Given ε take $M \geq \frac{1}{\sqrt{\frac{2}{\pi}\cdot\varepsilon}}$ then if $n > M$ $\frac{|\tan^{-1}(nx)|}{n^2} < \frac{\frac{\pi}{2}}{M^2} \leq \frac{\pi/2}{\frac{1}{\frac{2}{\pi}\varepsilon}}$

$= \varepsilon$

c) Let $U_n(x) = x^n$ then $|U_n(x)| \leq \left(\frac{3}{4}\right)^n$ and by the Weierstrass M test $\sum_{n=1}^{\infty} x^n$ converges uniformily on $[-3/4, 1/2]$

8. Show that the function $f(x) = x^3 + x^2$, regarded as defined on the closed interval $[1, 2]$, has a continuous inverse defined on the interval $[2, 12]$. You may use basic information from differential calculus.

Recall that if $f\ I \to \mathbb{R}$ is strictly monotone then $f^{-1}\ f(I) \to \mathbb{R}$ is continuous

if $1 < x_1 < x_2$ then $x_1^2 < x_2^2$ and $x_1^3 < x_2^3$ so $x_1^3 + x_1^2 < x_2^3 + x_2^2$, so f is strictly monotone,

$f(1) = 2$, $f(2) = 12$ so $f([1,2]) = [2, 12]$

$f^{-1}\ [2, 12] \to \mathbb{R}$ is therefore continuous