MIDTERM #1

Math 327A

name

You must show all work for full credit. Use the backs of the test pages as necessary.

1. Show carefully, using the definition of limit, that the sequence $\{s_n\}$ defined by $s_n = \sin n/\sqrt{n}$ converges. What is its limit?

The limit is 0 so we want to show $\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \forall n \geq M$
$\left|\frac{\sin n}{\sqrt{n}}\right| < \varepsilon$

Given ε, choose $M > \frac{1}{\varepsilon^2}$ then if $n \geq M$ $\left|\frac{\sin(n)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \leq$

$\leq \frac{1}{\sqrt{M}} \leq \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon$

2. The following three assertion are *incorrect* statements of theorems discussed in class. In each case, give the *correct* statement of the theorem.

a. Every increasing or decreasing sequence is convergent. Add bounded
b. Every sequence has a convergent subsequence. Add bounded
c. Every set of real numbers that is bounded below has a ~~smallest~~ greatest lower bound.

3. Give three examples of sequences of real numbers, one with only one number occurring as the limit of a subsequence, another with two such numbers, and the last one with three such numbers.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$1, 1+\frac{1}{2}, \frac{1}{2}, 1+\frac{1}{3}, \frac{1}{3}, 1+\frac{1}{4}, \frac{1}{4}, \ldots$ (some subsequences do not converge)

$1, 1+\frac{1}{2}, 2+\frac{1}{2}, \frac{1}{2}, 1+\frac{1}{2}, 2+\frac{1}{2}, \frac{1}{3}, 1+\frac{1}{3}, 2+\frac{1}{3}, \frac{1}{4}, 1+\frac{1}{4}, 2+\frac{1}{4}, \frac{1}{5}, 1+\frac{1}{5}, 2+\frac{1}{5} \ldots$

Question: is it true that if all subsequences of $\{a_n\}$ converge, they must converge to the same number a so $a_n \to a$ as well?

4. Define a sequence $\{s_n\}$ of real numbers inductively via $s_1 = 1, s_{n+1} = \sqrt{s_n + 2}$. Show that this sequence converges and evaluate its limit.

$\forall n \in N$ $S_n \leq S_{n+1}$ and $S_n \leq 2$ by induction

1) Base case: $n=1$ $s_1 = 1 \leq s_2 = \sqrt{1+2}$, and $1 \leq 2$

2) Induction step: assume $S_n \leq S_{n+1}$ and $S_n \leq 2$

then $S_{n+1} = \sqrt{S_n + 2} \leq \sqrt{S_{n+1} + 2} = S_{n+2}$ and $S_{n+1} = \sqrt{S_n + 2} \leq \sqrt{2+2} = 2$

So S_n is bounded and increasing so it must converge

if $\lim_{n \to +\infty} S_n = \ell$ then $\lim_{n \to +\infty} S_{n+1} = \ell$; $\lim_{n \to +\infty} S_{n+1} = \lim_{n \to +\infty} \sqrt{S_n + 2}$

gives $\ell = \sqrt{\ell + 2}$ so ℓ is a solution of $\ell^2 - \ell - 2 = 0$ $\ell = 2$ or $\ell = -1$

clearly $\ell \geq 0$ since $S_n \geq 0$ so $\ell = 2$

5. Let $\{s_n\}$ be a convergent sequence of real numbers such that $0 \leq s_n \leq 1$ for all n. Show that $\lim_{n\to\infty} s_n$ lies between 0 and 1.

Let $\ell = \lim_{n \to +\infty} S_n$ then given $\varepsilon > 0$ $\exists M \in N$ $\forall n \geq M$

$| S_n - \ell | < \varepsilon$ so $-\varepsilon < S_n - \ell < \varepsilon$ $\ell < S_n + \varepsilon \leq 1 + \varepsilon$

and $\ell > S_n - \varepsilon \geq -\varepsilon$ since the above inequalities hold for every $\varepsilon > 0$ we must have $\ell \leq 1$ and $\ell \geq 0$