MATH 327 A
SPRING 2014
FINAL EXAM ANSWERS

1. (15 points) Determine whether the series converges absolutely, converges conditionally or diverges. Justify your answer.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k+1} \cdot k^2}{3^k}$

ANSWER: For each k, let $a_k = \frac{2^{k+1} \cdot k^2}{3^k}$. Then,

$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \to \infty} \frac{2}{3} \left(1 + \frac{1}{k}\right)^2 = \frac{2}{3} < 1.$$

By the Ratio Test, $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k+1} \cdot k^2}{3^k}$ converges absolutely.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{4k^{1/5}}{4k-3}$

ANSWER: For all k,

$$\frac{4k^{1/5}}{4k-3} > \frac{4k^{1/5}}{4k} = \frac{1}{k^{4/5}}.$$

The series $\displaystyle\sum \frac{1}{k^{4/5}}$ is a p-series with $p = \frac{4}{5} < 1$. So, $\displaystyle\sum \frac{1}{k^{4/5}}$ diverges and thus $\displaystyle\sum \frac{4k^{1/5}}{4k-3}$ diverges by the Comparison Test.

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \cdot \frac{k+1}{k^2}$

ANSWER: We first check for absolute convergence. For all k,

$$\left|(-1)^k \cdot \frac{k+1}{k^2}\right| = \frac{k+1}{k^2} = \frac{1}{k} + \frac{1}{k^2} > \frac{1}{k}.$$

Since $\displaystyle\sum \frac{1}{k}$ diverges, $\displaystyle\sum \frac{1}{k} + \frac{1}{k^2}$ diverges by the Comparison Test and thus $\displaystyle\sum_{k=1}^{\infty} (-1)^k \cdot \frac{k+1}{k^2}$ does not converge absolutely.

For each k, let $a_k = \frac{k+1}{k^2} = \frac{1}{k} + \frac{1}{k^2}$. Then, for each k, $a_k > 0$ and $\lim_{k \to \infty} a_k = 0$. Moreover, since for each k, $0 \le k \le k+1$ implies $0 \le k^2 \le (k+1)^2$, we have $\frac{1}{k+1} \le \frac{1}{k}$, $\frac{1}{(k+1)^2} \le \frac{1}{k^2}$ and thus $a_{k+1} = \frac{1}{k+1} + \frac{1}{(k+1)^2} \le \frac{1}{k} + \frac{1}{k^2} = a_k$. By the Alternating Series Test, $\displaystyle\sum_{k=1}^{\infty} (-1)^k \cdot \frac{k+1}{k^2}$ converges. Since $\displaystyle\sum_{k=1}^{\infty} (-1)^k \cdot \frac{k+1}{k^2}$ converges but does not converge absolutely, it converges conditionally.

2. (15 points) For each $n \in \mathbb{N}$, define $f : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{x^n}{1+x^n}.$$

(a) Determine the function f to which $\{f_n\}$ converges pointwise on $[0,1]$

ANSWER: If $0 \le x < 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0.$$

If $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+1} = \frac{1}{2}.$$

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } x = 1. \end{cases}$$

Then $\{f_n\}$ converges pointwise to f on $[0,1]$.

(b) Is convergence uniform on $[0,1]$? Prove you are correct.

ANSWER: No, convergence is not uniform on $[0,1]$. For each n, $f_n(x)$ is continuous on $[0,1]$ (since it is a rational function that is never 0 on this interval) but $f(x)$ is not continuous at 1. By Theorem 5 on the list of Results, $\{f_n\}$ does not converge uniformly to f on $[0,1]$.

(c) Suppose $0 < r < 1$. Is convergence uniform on $[0,r]$? Prove you are correct.

ANSWER: Yes. For each $n \ge 1$,

$$|f_n(x) - 0| = \frac{x^n}{1+x^n} \le x^n \le r^n \text{ for all } x \in [0,r].$$

Since $0 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$ and by the Comparison Lemma for Uniform Convergence, $\{f_n\}$ converges uniformly to f on $[0,1]$.

3. (11 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x < 0 \\ 5 & \text{if } x = 0 \\ 7x - 1 & \text{if } x > 0. \end{cases}$$

Determine $\lim\limits_{x\to 0} f(x)$ and use the ϵ-δ definition of the limit to prove you are correct.

Claim: $\lim\limits_{x\to 0} = -1$.

Proof: If $x < 0$, then

$$|f(x) - (-1)| = \left| \frac{1}{x-1} + 1 \right| = \frac{|x|}{|x-1|}.$$

Still assuming $x < 0$, we have $x - 1 < -1 < 0$, which implies that $|x - 1| = -(x - 1) > 1 > 0$. This gives $\dfrac{1}{|x-1|} < 1$, which means that, for $x < 0$, $|f(x) - (-1)| = \dfrac{|x|}{|x-1|} < |x|$.

On the other hand, if $x > 0$, then

$$|f(x) - (-1)| = |7x - 1 + 1| = 7|x|.$$

Suppose $\epsilon > 0$ and let $\delta = \frac{\epsilon}{7}$. If $0 < |x - 0| < \delta$, then either $|f(x) - (-1)| < |x| < \delta < \epsilon$ or $|f(x) - (-1)| < 7|x| < 7\delta = \epsilon$. Thus, $\lim_{x \to 0} f(x) = -1$. $\square$

4. (15 points) Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \frac{x + |x|}{2}.$$
Determine the values of x at which f is continuous. Prove you are correct.

ANSWER: Note that, by definition of absolute value,
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Claim: f is continuous on $\mathbb{R}$.

Proof: Suppose $x_0 < 0$ and let $\{x_n\}$ be a sequence that converges to x_0. By Theorem 1 on the Results, there is an N such that, if $n \geq N$, then $x_n < 0$. For all $n \geq N$, we then have $f(x_n) = 0$ and thus, $\lim_{n\to\infty} f(x_n) = 0 = f(x_0)$. Thus, f is continuous at x_0 in the case that $x_0 < 0$.

Suppose instead that $x_0 > 0$ and let $\{x_n\}$ be a sequence that converges to x_0. Again by Theorem 1, there is an N_1 such that, if $n \geq N_1$, then $x_n > 0$. Suppose $\epsilon > 0$. Since $\{x_n\}$ converges to x_0, there is an N_2 such that, if $n \geq N_2$, then $|x_n - x_0| < \epsilon$. Suppose $n \geq \max\{N_1, N_2\}$. Then $|f(x_n) - f(x_0)| = |x_n - x_0| < \epsilon$. Thus, $\lim_{n\to\infty} f(x_n) = f(x_0)$ and f is continuous at x_0 in the case that $x_0 > 0$.

Finally, let $\{x_n\}$ be a sequence that converges to 0 and suppose $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = 0$, there is an N such that, if $n \geq N$, then $|x_n - 0| < \epsilon$. Suppose $n \geq N$. If $x_n < 0$, then $|f(x_n) - f(0)| = |0 - 0| = 0 < \epsilon$. On the other hand, if $x_n \geq 0$, then $|f(x_n) - f(0)| = |x_n - 0| < \epsilon$. Thus, $\lim_{n\to\infty} f(x_n) = 0 = f(0)$ and f is continuous at 0.

Since f is continuous at every $x_0 \in \mathbb{R}$, f is continuous on $\mathbb{R}$. □