Math 327 Spring 2017 Midterm 2 Exam

Write clearly and legibly. Justify all your answers.
You will be graded for correctness and clarity of your solutions.
You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.
You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.
Please raise your hand and ask a question if anything is not clear.
This exam contains 6 pages and is worth a total of 50 points. You have 50 minutes. Good luck

NAME:_________

PROBLEM 1 (20 points) _______

PROBLEM 2 (15 points) _______

PROBLEM 3 (15 points) ________

Total (50 points) ____________

- **Problem 1** (20 points) Decide whether the following series are absolutely convergent, conditionally convergent (that is convergent, but not absolutely convergent) or divergent. Remember to justify your answer.

  a) $\sum_{i=1}^{\infty}(-1)^i \frac{5i}{2i+3}$

$\lim_{i \to +\infty} \frac{5i}{2i+3} = \frac{5}{2+3/i} = \frac{5}{2}$ therefore the sequence

$\{a_i = (-1)^i \frac{5i}{2i+3}\}$ has subsequences $\{a_{2i}\}$ converging to $\frac{5}{2}$

and $\{a_{2i-1}\}$ converging to $-\frac{5}{2}$, therefore it diverges

so it does not converge to 0 therefore $\sum_{i=1}^{\infty} a_i$

diverges.

  b) $\sum_{i=1}^{\infty} \sin(i) \frac{2^i}{(i-1)!}$

First consider $\sum_{i=1}^{\infty} \frac{2^i}{(i-1)!}$ this series converges

by ratio test since $\lim_{i \to +\infty} \frac{2^{i+1}}{i!} \cdot \frac{(i-1)!}{2^i} = \lim_{i \to +\infty} \frac{2}{i} = 0$

$\left| \sin(i) \frac{2^i}{(i-1)!} \right| \leq \frac{2^i}{(i-1)!}$ therefore by comparison test

$\sum_{i=1}^{\infty} \sin(i) \frac{2^i}{(i-1)!}$ is absolutely convergent

(problem 1 continued)

c) $\sum_{i=1}^{\infty}(-1)^i \frac{3}{2^i}$

$\sum_{i=1}^{\infty} \frac{1}{2^i}$ is a geometric series that converges by a th proved in class $\sum_{i=1}^{\infty} \frac{3}{2^i}$ converges

Therefore $\sum_{i=1}^{\infty} (-1)^i \frac{3}{2^i}$ is absolutely convergent.

d) $\sum_{i=1}^{\infty} \frac{(-2)^i}{2^i(i+1)} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i+1}$ is not absolutely convergent since $\lim_{i \to +\infty} \frac{\frac{1}{i+1}}{\frac{1}{i}} = \frac{i}{i+1} = \frac{1}{1+\frac{1}{i}} = 1$ so $\sum_{i=1}^{\infty} \frac{1}{i+1}$ diverges

but $\sum_{i=1}^{\infty} \frac{(-1)^i}{i+1}$ converges by alternating series test since $\lim_{i \to +\infty} \frac{1}{i+1} = 0$ and $\frac{1}{i+1+1} < \frac{1}{i+1}$

- **Problem 2** (15 points) For each of the following series determine all the values of $p \in R$ for which the series converges. Remember to justify your answer.

  a) $\sum_{i=1}^{\infty} \frac{p^i}{5^i}$

This is a geometric series. It converges if $\left|\frac{p}{5}\right| < 1$

so $|p| < 5$ or $-5 < p < 5$

  b) $\sum_{i=1}^{\infty} \frac{i^p}{\sqrt{i^4+1}}$

$$\lim_{i \to +\infty} \frac{\frac{i^p}{\sqrt{i^4+1}}}{\frac{1}{i^{2-p}}} = \frac{i^p}{i^2\sqrt{1+1/i^4}} \cdot i^{2-p} = 1$$

Therefore the series behaves like $\sum_{i=1}^{\infty} \frac{1}{i^{2-p}}$

and this converges if $2-p > 1$ so $p < 1$

  c) $\sum_{i=1}^{\infty} \frac{p^i}{(2i)!}$

$$\lim_{i \to +\infty} \frac{|p|^{i+1}}{(2i+2)!} \cdot \frac{(2i)!}{|p|^i} = \frac{|p|}{(2i+1)(2i+2)} = 0$$

converges for all values of p

- **Problem 3** In this problem we shall consider two series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ with $0 \leqslant a_i$ ~~$\leqslant b_i$~~ for all $i \in N$.
  $0 < b_i$

  a) Prove that if $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$ and $\sum_{i=1}^{\infty} b_i$ converges then $\sum_{i=1}^{\infty} a_i$ converges

Given $\varepsilon = 1$ there is M s.t $\forall i \geqslant M$ $\frac{a_i}{b_i} < 1$

so $a_i < b_i$ $\forall i \geqslant M$

and therefore by the comparison test $\sum_{i=1}^{\infty} a_i$ converges.

  b) Give an examples of series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ with $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$ and $\sum_{i=1}^{\infty} a_i$ convergent, but $\sum_{i=1}^{\infty} b_i$ divergent

$a_i = \frac{1}{i^2}$ $b_i = \frac{1}{i}$

  c) If $\lim_{i\to\infty} \frac{a_i}{b_i} = \boxed{+\infty}$ then if $\sum_{i=1}^{\infty} a_i$ converges then $\sum_{i=1}^{\infty} b_i$ converges, but the converse of this statement is not true, that is it is possible to find series $\sum_{i=1}^{\infty} a_i$ that diverges and $\sum_{i=1}^{\infty} b_i$ that converges , with $\lim_{i\to\infty} \frac{a_i}{b_i} = \boxed{+\infty}$. Enter something in the box that makes the above statement true and prove it.

Given $K = 1$ $\exists M \in N$ $\forall i \geqslant M$ $\frac{a_i}{b_i} > 1$ so $a_i > b_i$

so again by comparison test $\sum_{i=1}^{\infty} b_i$ converges

comparison th $\sum_{i=1} b_i$ converges.

Again if $a_i = \frac{1}{i}$ $b_i = \frac{1}{i^2}$ $\lim_{i\to+\infty} \frac{a_i}{b_i} = +\infty$

$\sum_{i=1} b_i$ converges, $\sum_{i=1}^{\infty} a_i$ diverges