- **Problem 1** ($\cancel{10}$ 8 points) Find $\lim_{n\to\infty} \frac{3n+5}{n+7}$ and prove your result.

$$\lim_{n\to+\infty} \frac{3n+5}{n+7} = 3$$

Proof: $\frac{3n+5}{n+7} = \frac{n(3+5/n)}{n(1+7/n)}$ and $\lim_{n\to+\infty} \frac{3+5\cdot 1/n}{1+7\cdot 1/n} = 3$

by the limit laws and the fact that $\lim_{n\to+\infty} \frac{1}{n} = 0$

Alternative proof

We need to prove $\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \forall n \geq M\ \left|\frac{3n+5}{n+7} - 3\right| < \varepsilon$

Given $\varepsilon$ take $M > \frac{16-7\varepsilon}{\varepsilon}$ then if $n > M$

$n > \frac{16-7\varepsilon}{\varepsilon}$ so $16 < n\varepsilon + 7\varepsilon$ so $\frac{16}{n+7} < \varepsilon$ so $\left|\frac{-16}{n+7}\right| < \varepsilon$ so

$\left|\frac{3n+5-3n-21}{n+7}\right| < \varepsilon$ so $\left|\frac{3n+5}{n+7} - 3\right| < \varepsilon$

Scratchwork: $\left|\frac{3n+5}{n+7} - 3\right| < \varepsilon \iff$

$\left|\frac{3n+5-3n-21}{n+7}\right| < \varepsilon \iff$

$\left|\frac{-16}{n+7}\right| < \varepsilon \iff$

$\frac{16}{n+7} < \varepsilon \iff$

$16 < \varepsilon n + 7\varepsilon \iff$

$\frac{16-7\varepsilon}{\varepsilon} > n$

S conteins $\frac{1}{2}-1, \frac{3}{4}-1, \frac{5}{6}-1, \ldots$

$\frac{2}{3}+1, \frac{4}{5}+1, \frac{6}{7}+1, \ldots$

(16 points)

- **Problem 2** Given $S = \{\frac{n}{n+1} + (-1)^n | n \in N\}$ Let $s = \sup S$ and $i = \inf S$

a) ~~(6 points)~~ Find the value of $i$ and prove $i = \inf S$.

$i = -\frac{1}{2}$.

Proof :

1) If $n$ is odd, $-\frac{1}{2} \le \frac{n}{n+1} - 1 \iff \frac{1}{2} \le \frac{n}{n+1} \iff n+1 \le 2n$

$\iff 1 \le n$, True

If $n$ is even, $-\frac{1}{2} \le \frac{n}{n+1} + 1 \iff -\frac{3}{2} \le \frac{n}{n+1} \iff -3n-3 \le 2n \iff$

$-\frac{3}{5} \le n$, True. So $-\frac{1}{2}$ is a lower bound for $S$

$-\frac{1}{2} = \frac{1}{1+1} - 1$ so $-\frac{1}{2} \in S$ therefore $-\frac{1}{2} = \min(S)$

b) ([redacted]ts) Find the value of $s$ and prove $s = \sup S$

$\sup(S) = 2$ proof :

1) If $n$ is odd, $\frac{n}{n+1} - 1 \leq 2 \iff n \leq 3n+3 \iff -3 \leq 2n \iff -\frac{3}{2} \leq n$ True

If $n$ is even $\frac{n}{n+1} + 1 \leq 2 \iff n \leq n+1 \iff 0 \leq 1$ True

2) Given $\varepsilon > 0$ we can find $n$ even s.t $\frac{n}{n+1} + 1 > 2 - \varepsilon$ since

$\frac{n}{n+1} + 1 > 2 - \varepsilon \iff n \geq (n+1)(1-\varepsilon) \iff n \geq n - \varepsilon n + (1-\varepsilon) \iff$

$\varepsilon n \geq (1-\varepsilon) \iff n > \frac{1-\varepsilon}{\varepsilon}$ and by the Archimedean property

we can find an even $n$ st $n > \frac{1-\varepsilon}{\varepsilon}$

c) ([redacted]s) Is $S$ closed ? Justify your answer.

No take $a_n = \frac{2n}{2n+1} + 1$ then $a_n \in S$ $a_n \to 2$

but $2 \notin S$ since if $n$ is odd $\frac{n}{n+1} - 1 = 2 \iff \frac{n}{n+1} = 3$

$\iff n = 3n+1 \iff n = -1/2$ impossible. If $n$ is even

$\frac{n}{n+1} + 1 = 2 \iff n = n+1 \iff 0 = 1$ impossible

d)([redacted]s) Is $S$ sequentially compact ? Justify your answer.

No because it is not closed

- **Problem 3**(~~11~~ 8 points) Let $\{a_n\}$ be a convergent sequence. Prove that $\{(-1)^n a_n\}$ converges if and only if $\{a_n\}$ converges to 0.

$\Leftarrow$ Assume $\lim_{n\to+\infty} a_n = 0$ we need to prove $\{(-1)^n a_n\}$ converges

(we shall actually prove $\lim_{n\to\infty} (-1)^n a_n = 0$):

Since $|(-1)^n a_n| = |a_n|$ we have that

$\forall \varepsilon > 0 \; \exists M \in \mathbb{N} \; \forall n \geq M \; |a_n| < \varepsilon \Rightarrow \forall \varepsilon > 0 \; \exists M \in \mathbb{N} \; \forall n \geq M \; |(-1)^n a_n| < \varepsilon$

Alternatively use the squeeze thm $\underbrace{-a_n}_{\to 0} \leq (-1)^n a_n \leq \underbrace{a_n}_{\to 0}$

Note: you cannot use the limit laws to say

$\lim_{n\to+\infty} (-1)^n a_n = \lim_{n\to+\infty} (-1)^n \lim_{n\to+\infty} a_n$ because $\{(-1)^n\}$

diverges.

$\Leftarrow$ (bottom of the page)

. Can $\{(-1)^n a_n\}$ converge if $\{a_n\}$ diverges ? ~~Prove~~ Justify your answer.

yes if $a_n = (-1)^n$ then $\{a_n\}$ diverges

but $(-1)^n \cdot a_n = (-1)^n (-1)^n = 1$ and the constant

sequence equal to 1 converges.

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$\Leftarrow$ by contraposition assume $\{a_n\}$ converges to $a \neq 0$

then $\{(-1)^n a_n\}$ has two subsequences : $\{(-1)^{2k} a_{2k}\}$ converging

to $a$ and $\{(-1)^{2k+1} a_{2k+1}\}$ converging to $-a$, so

$\{(-1)^n a_n\}$ does not converge.

- **Problem 4**(12 points) Give examples for each of the following, or explain why it is not possible to provide an example

1. A subset $S$ of $R$ that is closed and bounded, but it is not an interval.

$[1,2] \cup [3,4]$

2. Two convergent sequences $\{a_n\}$ and $\{b_n\}$ with $b_n \neq 0$, for all $n \geq 1$ such that their quotient $\{\frac{a_n}{b_n}\}$diverges.

$a_n = 1 \qquad b_n = \frac{1}{n}$

Note that $\lim_{n \to +\infty} \frac{1}{n} = 0$

3. A convergent sequence $\{a_n\}$ such that the set $S = \{a_n\,; n \in N\}$ has no greatest lower bound.

Impossible $\{a_n\}$ is bounded so $S$ is bounded below

4. A bounded sequence that is not convergent.

$a_n = (-1)^n$

5. A closed subset of R that is not sequentially compact.

$[0, +\infty)$

6. A Cauchy sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = \infty$

Impossible, a cauchy sequence is convergent