Math 327 Spring 2017 Final Exam

Write clearly and legibly. Justify all your answers.

You will be graded for correctness and clarity of your solutions.

You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.

You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.

Please raise your hand and ask a question if anything is not clear.

This exam contains 8 pages and is worth a total of 80 points. You have 1 hour and 50 minutes. Good luck

NAME:_________

PROBLEM 1 (20 points) _______

PROBLEM 2 (10 points) _______

PROBLEM 3 (15 points) ________

PROBLEM 4 (15 points)________

PROBLEM 5 (10 points) ________

PROBLEM 6 (10 points) ________

Total ____________

- **Problem 1** (20 points) Consider the following sequence $\{a_n\}$ and answer the questions below. Remember to justify your answers.

$$a_n = \begin{cases} \frac{n+1}{n+2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

a) Is $\{a_n\}$ bounded ?

yes

$\forall n \in N \quad |a_n| < 1$

b) Is $\{a_n\}$ monotone ?

NO $\quad a_1 = 0, \; a_2 = \frac{3}{4}, \; a_3 = 0$

so $a_1 \leq a_2$ but
$a_2 \geq a_3$

c) Is $\{a_n\}$ convergent ?

No $\quad a_{2n-1} \to 0$
$a_{2n} = \frac{2n+1}{2n+2} = \frac{n(2 + 1/n)}{n(2 + \frac{2}{n})} \to 1$

d) Is $\{a_{2n}\}$ convergent ? Give a proof using the definition of lim of a sequence

yes $\quad a_{2n} \to 1$

we need to prove $\forall \varepsilon > 0 \; \exists M \; \forall n \geq M \; \left| \frac{2n+1}{2n+2} - 1 \right| < \varepsilon$

given ε take $M > \frac{1}{2\varepsilon} - 1$ then if $n > M \quad n > \frac{1}{2\varepsilon} - 1 \Rightarrow$

$n+1 > \frac{1}{2\varepsilon} \Rightarrow \frac{1}{2(n+1)} < \varepsilon \Rightarrow \left| \frac{-1}{2n+2} \right| < \varepsilon \Rightarrow \left| \frac{2n+1-2n-2}{2n+2} \right| < \varepsilon$

$\Rightarrow \left| \frac{2n+1}{2n+2} - 1 \right| < \varepsilon$

(PROBLEM 1 CONTINUED)

e) Let $S = \{a_n | \quad n \in N\}$ find s=sup S and i= inf S and prove s= sup S and i= inf S.

$s = 1$. If n is odd $0 \leq 1$
If n is even $\frac{n+1}{n+2} \leq 1$ so 1 is an upper bound for S.

Since $a_{2n} \to 1$ given $\varepsilon > 0$ $\exists M$ $\forall n \geq M$ $|a_{2n} - 1| < \varepsilon$

so take any $n > M$ $1-\varepsilon < a_{2n}$

$i = 0$ if n is odd $a_n = 0 \leq 0$
if n is even $0 \leq \frac{n+1}{n+2}$

$0 \in S$ so $0 = \min S$

- **Problem 2** (10 points) Prove that if $f : D \to R$ and $g : D \to R$ are both uniformly continuous, then their sum is uniformly continuous.

We know that given ε

1) $\exists \delta_1 > 0 \ \forall x, y \in D, \ |x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon/2$ and

2) $\exists \delta_2 > 0 \ \forall x, y \in D, \ |x-y| < \delta_2 \Rightarrow |g(x) - g(y)| < \varepsilon/2$

We want to show: $\forall \varepsilon \ \exists \delta \ \forall x, y \in D, \ |x-y| < \delta \Rightarrow$

$$|(f(x)+g(x)) - (f(y)+g(y))| < \varepsilon$$

given ε let δ_1 and δ_2 as above

take $\delta = \min(\delta_1, \delta_2)$ then if $|x-y| < \delta$ we have

$|(f(x)+g(x)) - (f(y)+g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon$

Alternatively

Given $\{u_n\}, \{v_n\} \in D$ s.t $(u_n - v_n) \to 0$ We know $(f(u_n) - f(v_n)) \to 0$ $(g(u_n) - g(v_n)) \to 0$ because f and g are uniformly continuous, therefore $(f(u_n) + g(u_n)) - (f(v_n) + g(v_n)) = (f(u_n) - f(v_n)) + (g(u_n) - g(v_n)) \to 0$

- **Problem 3** (15 points) For each of the following either give a proof (quoting a theorem from class without proof is OK) or a counterexample.

a) If $f : D \to R$ is continuous and D is open then $f(D)$ is open.

Counterexample: $f\ (0, 2\pi) \to R$
$f(x) = \sin x$
$f(D) = [-1, 1]$

b) If $f : D \to R$ is continuous and D is a closed and bounded interval (that is an interval of the form $[a, b]$) then $f(D)$ is a closed and bounded interval.

True, by the intermediate value th $f(D)$ is an interval.
By the extreme value th. $f(D)$ has max and min.
so $f(D) = [c, d]$ for some $c, d \in R$

c) If $f : D \to R$ is continuous and D is bounded then $f(D)$ is bounded.

Counterexample $f\ (0, 1) \to R$
$f(x) = \frac{1}{x}$
$f(D) = (0, +\infty)$

d) If $f : D \to R$ is uniformily continuous and D is bounded then $f(D)$ is bounded.

True this is a th we proved in class

- **Problem 4** (15 points) Given $f : R \to R$ defined as follows :

$$\begin{cases} f(x) = 0 & \text{if} \quad x = 0 \\ \frac{1}{x} & \text{if} \quad x \in Q \quad \text{but} \quad x \neq 0 \\ x & \text{if} \quad x \notin Q \end{cases}$$

a) Is f continuous at 0 ? Justify your answer.

No let $x_n = \frac{1}{n}$ then $x_n \to 0$ $x_n \in Q$ $x_n \neq 0$
so $f(x_n) = n$ and $f(x_n)$ does not converge to
$f(0) = 0$

b) Is f continuous at $\sqrt{2}$? Justify your answer.

No. let $y_n \in Q$ $y_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n})$, (we can find y_n since Q is dense in R) $y_n \neq 0$ $y_n \in Q$ $y_n \to \sqrt{2}$
so $f(y_n) = \frac{1}{y_n} \to \frac{1}{\sqrt{2}} \neq f(\sqrt{2}) = \sqrt{2}$

c) Is f continuous at 1 ? Justify your answer.

Yes suppose $x_n \to 1$ we want to show
$f(x_n) \to 1$. Take $\varepsilon = \frac{1}{2}$ we know $\exists M_1$ $\forall n \geq M$
$|x_n - 1| < \frac{1}{2}$ or $\frac{1}{2} < x_n < \frac{3}{2}$ (This just to show $x_n \neq 0$ and bounded below)

We also know given ε, $\exists M_2$ $\forall n \geq M_2$ $|x_n - 1| < \varepsilon \cdot \frac{1}{2}$,
take $M = \max(M_1, M_2)$ if $n > M$ and $f(x_n) = x_n$ we have
$|f(x_n) - 1| = |x_n - 1| < \frac{\varepsilon}{2} < \varepsilon$, if $f(x_n) = \frac{1}{x_n}$ we have
$|f(x_n) - 1| = \left| \frac{1}{x_n} - 1 \right| = \left| \frac{x_n - 1}{x_n} \right| \leq \frac{|x_n - 1|}{\frac{1}{2}} \leq 2|x_n - 1| < 2\frac{\varepsilon}{2} = \varepsilon$

Alternative proof of c)

We need to prove that

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \quad |x-1| < \delta \Rightarrow |f(x) - 1| < \varepsilon$

Given ε take $\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$

if $|x-1| < \delta$ and $x \notin \mathbb{Q}$

$$|f(x) - 1| = |x-1| < \frac{\varepsilon}{2} < \varepsilon$$

if $x \in \mathbb{Q}$ $\quad 1 - \frac{\varepsilon}{2} < x < 1 + \frac{\varepsilon}{2}$

and $\quad \frac{1}{2} < x < \frac{3}{2}$ $\quad$ so $x \neq 0$

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right|$$

$$\leq \frac{|x-1|}{\frac{1}{2}} \leq 2\,\frac{\varepsilon}{2} = \varepsilon$$

- **Problem 5** (10 points) Prove that given $M \in N$, $, M > 1$, $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sum_{i=M}^{\infty} a_i$ converges.(Hint: Use sequences of partial sums).

Let $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=M}^{n} a_i$ for $n \geq M$

then $s_n = (a_1 + \cdots + a_{M-1}) + t_n$

and $t_n = s_n - (a_1 + \cdots + a_{M-1})$, $a_1 + \cdots + a_{M-1}$ is just a constant value.

by limit theorems on sequences

if $s_n \to \alpha$ $t_n \to \alpha + (a_1 + \cdots + a_{M-1})$ so if $\sum_{i=1}^{\infty} a_i$ converges

then $\sum_{i=M}^{\infty} a_i$ converges.

if $t_n \to \beta$ then $s_n \to \beta - (a_1 + \cdots + a_{M-1})$

so if $\sum_{i=M}^{\infty} a_i$ converges then $\sum_{i=1}^{\infty} a_i$ converges

- **Problem 6** (10 points) Find the function f the sequence

$$f_n : R \to R$$

$$f_n(x) = \frac{x}{n}$$

converges to and prove the convergence is not uniform.

$\lim_{n \to +\infty} \frac{x}{n} = 0$ so f $R \to R$, $f(x) = 0$

We need to show that it is not true that
$\forall \varepsilon > 0 \ \exists M \in N \ \forall n \geq M \ \forall x \in R \ |f_n(x)| < \varepsilon$, that is
we need to show
$\exists \varepsilon > 0 \ \forall M \in N \ \exists n \geq M \ \exists x \in R \ |f_n(x)| > \varepsilon$

Take $\varepsilon = 1$, given M, choose $n = M$ and $x = M+1$

then $\frac{x}{n} = \frac{M+1}{M} > 1$

scratch work: we want $|\frac{x}{n}| > \varepsilon$ for some arbitrarily big n
and some $x \in R$ so (if $x>0$) $x > \frac{n}{\varepsilon}$ we can find
such an x no matter what ε is