Math 327 Fall 2016 Midterm 1

Write clearly and legibly. Justify all your answers.
You will be graded for correctness and clarity of your solutions.
You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.
You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.
Please raise your hand and ask a question if anything is not clear.
This exam contains 6 pages and is worth a total of 50 points. You have 50 minutes. Good luck

NAME:_________

PROBLEM 1 (10 points) _______

PROBLEM 2 (17 points)_______

PROBLEM 3 (11 points) ________

PROBLEM 4 (12 points) ________

Total ____________

- **Problem 1** (10 points) Find $\lim_{n\to\infty} \frac{n+1}{2n+1}$ and prove your result.

$$\lim_{n\to+\infty} \frac{n+1}{2n+1} = \frac{1}{2}$$

Proof : $\frac{n+1}{2n+1} = \frac{n(1+1/n)}{n(2+\frac{1}{n})} = \frac{1+1/n}{2+\frac{1}{n}}$; using the fact that $\lim_{n\to+\infty} \frac{1}{n} = 0$ and limit laws we get $\lim_{n\to+\infty} \frac{n+1}{2n+1} = \frac{1+0}{2+0} = \frac{1}{2}$

Alternative proof using the limit definition

We need to prove that $\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \forall n \geq M\ \left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \varepsilon$

given ε, take $M > \frac{1}{2}\left(\frac{1}{2\varepsilon} - 1\right)$ then if $n \geq M$ $n > \frac{1}{2}\left(\frac{1}{2\varepsilon} - 1\right)$ so

$2n > \frac{1}{2\varepsilon} - 1$ so $2n+1 > \frac{1}{2\varepsilon}$ so $\frac{1}{2(2n+1)} < \varepsilon$ so

$\left| \frac{1}{2(2n+1)} \right| < \varepsilon$ so $\left| \frac{1}{2(2n+1)} + \frac{1}{2} - \frac{1}{2} \right| < \varepsilon$ and

$\left| \frac{2n+2}{2(2n+1)} - \frac{1}{2} \right| < \varepsilon$ so $\left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \varepsilon$

- **Problem 2** Given $S = \{\frac{n+1}{2n+1} | n \in N\}$ Let $s = \sup S$ and $i = \inf S$

a) (6 points) Find the value of i and prove $i = \inf S$. $i = 1/2$

Proof

1) $\frac{1}{2} \leq \frac{n+1}{2n+1}$ $\Leftrightarrow$

$2n+1 \leq 2n+1$ $\Leftrightarrow$

$1 \leq 1$ True $\forall n \in N$

2) Since we already proved in problem 1 that $\frac{n+1}{2n+1} \to \frac{1}{2}$
it is true that given ε $\exists M$ $\frac{1}{2}+\varepsilon > \frac{M+1}{2M+1}$

(In this particular case
it is true that $\forall n \geq M$ $\frac{1}{2}+\varepsilon > \frac{n+1}{2n+1}$ because
$a_n = \frac{n+1}{2n+1}$ is decreasing, so $\lim_{n\to+\infty} a_n = \inf\{a_n\}$

but this is not required
and in general not true, by the definition of inf.)

b) (6 points) Find the value of s and prove $s = \sup S$ $\quad s = \frac{2}{3}$

proof

1) $\frac{n+1}{2n+1} \le \frac{2}{3} \iff$

$3n+3 \le 4n+2 \iff$

$1 \le n$ True for all n

so $\frac{2}{3}$ is an upper bound for S.

when $n=1$ $\frac{1+1}{2\cdot 1+1} = \frac{2}{3}$ so $\frac{2}{3} \in S$

so $\frac{2}{3}$ is a max for S and therefore also $\sup S$

c) (3 points) Is S closed ? Justify your answer.

no $\frac{n+1}{2n+1} \to \frac{1}{2}$ but $\frac{1}{2} \notin S$ since

$\frac{n+1}{2n+1} = \frac{1}{2} \iff$

$2n+2 = 2n+1 \iff$

$2=1$ impossible

d) (2 points) Is S sequentially compact ? Justify your answer.

No, because it is not closed.

Alternatively

No any sequence $\{e_n\}$ in S is a subsequence of $\left\{\frac{n+1}{2n+1}\right\}$

so $\{e_n\}$ and all its subsequences also converge to $\frac{1}{2} \notin S$

- **Problem 3**(7 points) Prove that if the sequence $\{a_n\}$ converges to a then the sequence $\{|a_n|\}$ converges to $|a|$.(you can use the inequality $(|x-y| \geq ||x|-|y||)$

Assume $a_n \to a$, then we need to prove $|a_n| \to |a|$, that is

$\forall \varepsilon > 0 \ \exists M \in \mathbb{N} \ \forall n \geq M \ \ ||a_n| - |a|| < \varepsilon$

given ε we know there is M s.t $\forall n \geq M \ \ |a_n - a| < \varepsilon$

therefore if $n \geq M \ \ ||a_n| - |a|| \leq |a_n - a| < \varepsilon$

(the same M that works for $\{a_n\}$ also works for $\{|a_n|\}$)

(4 points) Is it true that if $\{|a_n|\}$ converges to $|a|$ then $\{a_n\}$ converges to a ? Justify your answer.

No Consider $\{(-1)^n\}$

$\{|(-1)^n|\}$ = is the constant sequence equal to 1

so $|(-1)^n| \to |1| = 1$ but

$(-1)^n$ does not converge to 1

- **Problem 4**(12 points) Say if each of the statements below is True or False (just write T or F next to each of them). No justification is necessary.

1. A convergent sequence must be monotone and bounded. F

   It must be bounded, not necessarily monotone
   For ex $\left\{\frac{(-1)^n}{n}\right\}$ is not monotone but converges

2. A decreasing sequence converges. F

   If it is also bounded. Ex $\{-n\}$ is decreasing but diverges

3. A set S can have a maximum but no least upper bound. F

   If $m = \max(S)$ then $m = \sup(S)$

4. Q is closed. F

   $\mathbb{Q}$ is dense in $\mathbb{R}$ so we can find a sequence of rational numbers converging for example to $\sqrt{2} \notin \mathbb{Q}$

5. Q is open. F

   $0 \in \mathbb{Q}$ but any interval $(-\varepsilon, \varepsilon)$ contains irrational numbers for example $\frac{\sqrt{2}}{n}$ if n is big enough

6. Assume $\{a_n\}$ converges , and $\{b_n\}$ is a sequence. Then $\{a_n \cdot b_n\}$ converges if and only if $\{b_n\}$ converges. F

   if $\{b_n\}$ converges then $\{a_n \cdot b_n\}$ also converges but it is possible for $\{b_n\}$ to diverge and $\{a_n \cdot b_n\}$ still converge. For example $a_n = \frac{1}{n}$ $b_n = (-1)^n$