Math 327 Fall 2016 Final Exam

Write clearly and legibly. Justify all your answers.
You will be graded for correctness and clarity of your solutions.
You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.
You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.
Please raise your hand and ask a question if anything is not clear.
This exam contains 8 pages and is worth a total of 90 points. You have 1 hour and 50 minutes. Good luck

NAME:_________

PROBLEM 1 _______

PROBLEM 2 _______

PROBLEM 3 ________

PROBLEM 4 ________

PROBLEM 5________

PROBLEM 6 ________

Total ____________

- **Problem 1** (10 points) Prove that if $|r| < 1$ then $\{r^n\}$ converges (do a full proof using the definition of limit of a sequence, do not just quote a result from class)

We want to prove $\lim_{n \to +\infty} r^n = 0$. that is

$\forall \varepsilon > 0 \; \exists M \; \forall n \geq M \quad |r^n| < \varepsilon$

if $r = 0$ then $r^n = 0$ so obviously $|r^n| = 0 < \varepsilon \quad \forall n \geq 1$

assume $r \neq 0$. Given ε take $M > \frac{\ln \varepsilon}{\ln |r|}$ then if $n \geq M$ $n > \frac{\ln \varepsilon}{\ln |r|}$ so

$n \ln |r| < \ln \varepsilon$ so $\ln |r|^n < \ln \varepsilon$ so $|r|^n < \varepsilon$ (since $\ln x$ is an increasing function)

Scretch work $\quad |r^n| < \varepsilon \iff$

$|r|^n < \varepsilon \iff$ if $r \neq 0$

$n \ln |r| < \ln \varepsilon \iff$

$n > \frac{\ln \varepsilon}{\ln |r|}$

- **Problem 2** (15 points) Decide if the following functions are uniformly continuous, and prove your answer.

  1. $f : (0, +\infty) \to R, \quad f(x) = \frac{1}{x^2}$

No: take $u_n = \frac{1}{n}$ $\quad v_n = \frac{1}{2n}$ then $u_n - v_n = \frac{1}{2n} \to 0$

but $f(u_n) - f(v_n) = n^2 - (2n)^2 = -3n^2 \not\to 0$

  2. $f : (\frac{1}{2}, 2) \to R, \quad f(x) = \frac{1}{x^2}$

yes we want to prove

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in (\frac{1}{2}, 2) \quad |x-y| < \delta \Rightarrow \left| \frac{1}{x^2} - \frac{1}{y^2} \right| < \varepsilon$

Given ε take $\delta < \frac{\varepsilon}{64}$ then $\left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \frac{|y^2 - x^2|}{x^2 y^2} = \frac{|x-y||x+y|}{x^2 y^2} \leq$

$\frac{\delta \cdot 4}{\frac{1}{4} \cdot \frac{1}{4}} = 64\delta < \varepsilon$

Scratchwork: want $\left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = \frac{|x+y||x-y|}{x^2 y^2} < \varepsilon$

and on $(\frac{1}{2}, 2)$ $|x+y| \leq 4$, $x^2 y^2 \geq \frac{1}{16}$ so we just need

$\frac{|x+y||x-y|}{x^2 y^2} \leq 4 \cdot 16 \delta < \varepsilon$ so $\delta < \frac{\varepsilon}{64}$

3. **Problem 3** (15 points) Prove that the image of a closed and bounded set under a continuous function is closed. That is, if $f : D \to R$ is continuous and D is closed and bounded, then$f(D)$ is closed.

Assume $\{y_n\}$ is a sepuence in $f(D)$ and $y_n \to y_0$
we need to show $y_0 \in f(D)$
Since $y_n \in f(D)$, $y_n = f(x_n)$ for some $x_n \in D$
The sepence $\{x_n\}$ is bounded (becouse D is bounded)
so it has a convergent subsepuence $x_{n_k} \to x_0$, $x_0 \in D$
(because D is closed) and $f(x_{n_k}) \to f(x_0)$ because
f is continuous. But $f(x_{n_k}) = y_{n_k}$ is a subsequence of
$\{y_n\}$ so it must also converge to y_0, therefore
$y_0 = f(x_0)$ so $y_0 \in f(D)$.

4. **Problem 4** (15 points) Given $f\, D \to R$ define what it means to say that $\lim_{x\to x_0} f(x) = l$. We have given two equivalent definitions, you can use either one of the two.

If $\{x_n\}$ is a sequence in D, with $x_n \neq x_0$, such that $x_n \to x_0$ then $f(x_n) \to \ell$

or

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D - \{x_n\} \; |x - x_0| < \delta \Rightarrow | f(x) - \ell | < \varepsilon$

Given $f\, D \to R$ define what it means to say the f is continuous at $x_0 \in D$.

$\lim_{x \to x_0} f(x) = f(x_0)$

Using the definition above, prove that $f : R \to R$, f(x)=2x+1 is continuous everywhere (do a full proof, do not just quote the result from class that sums and products of continuous functions are continuous. You can use limit laws , if you like).

Suppose $x_0 \in R$ and $\{x_n\}$ is a sepuence s.t $x_n \to x_0$, then by limit properties of sepuences $2x_n + 1 \to 2x_0 + 1 = f(x_0)$

5. **Problem 5** (15 points) Say whether the following series diverge or converge and justify your answer.

a) $\sum_{i=1}^{\infty} \frac{\sqrt{i^2+1}}{i^4+5}$

it converges since $\sum_{i=1}^{\infty} \frac{1}{i^3}$ converges and

$$\lim_{i \to +\infty} \frac{\frac{\sqrt{i^2+1}}{i^4+5}}{\frac{1}{i^3}} = \frac{i^3 \cdot i \sqrt{1+1/i^2}}{i^4\left(1+\frac{5}{i^4}\right)} = 1$$

b) $\sum_{i=1}^{\infty} \frac{10^i}{i!}$

it converges by the ratio test

$$\lim_{n \to +\infty} \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = 0$$

c) $\sum_{i=1}^{\infty} a_i$ where $\begin{cases} a_i = \frac{1}{i} & \text{if i is odd} \\ a_i = \frac{1}{i^2} & \text{if i is even} \end{cases}$

It diverges since $a_i \geq b_i \geq 0$ where $b_i = \begin{cases} \frac{1}{i} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$

and $\sum_{i=1}^{\infty} b_i = \sum_{k=1}^{\infty} \frac{1}{2k-1}$

(more precisely if $S_n = \sum_{i=1}^{n} b_i$ and $t_n = \sum_{i=1}^{n} \frac{1}{2i-1}$

then $t_n = S_{2n}$)

and $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ diverges by limit comparison test

with $\sum_{k=1}^{\infty} \frac{1}{k}$: $\lim_{k \to +\infty} \frac{\frac{1}{2k-1}}{\frac{1}{k}} = \frac{k}{k(2-\frac{1}{k})} = \frac{1}{2}$

(so t_n diverges and since a subsequence S_{2n} of S_n diverges then S_n diverges too)

so $\sum_{i=1}^{\infty} b_i$ diverges and therefore $\sum_{i=1}^{\infty} a_i$ diverges

6. **Problem 6** (20 points) Consider the sequence of functions $\{f_n\}$, with $f_n : R \to R$ defined by $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$

a) find the function f the sequence converges pointwise

$f \quad R \to R$

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1/2 & \text{if } x = \pm 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

since if $|x| < 1 \quad x^{2n} \to 0$

if $|x| = 1 \quad |x|^{2n} = 1$

if $|x| > 1 \quad \frac{x^{2n}}{x^{2n}(1 + (1/x^2)^n)} \to 1 \quad$ since $\frac{1}{x^2} < 1$

b) does $\{f_n\}$ converge to f uniformily ? Justify your answer

No f is not continuous

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c) If you change the domain of f_n from R to $(0, \frac{1}{2})$, find the function f the sequence converges pointwise .

$$f \quad (0, \tfrac{1}{2}) \to R$$
$$f(x) = 0$$

d) Does the sequence in part c) converge uniformly ? Justify your answer.

yes. We need to show

$\forall \varepsilon > 0 \quad \exists M \in N \quad \forall n \geq M \quad \forall x \quad |f_n(x)| < \varepsilon$

Given ε choose $n > \dfrac{\ln \varepsilon}{\ln \frac{1}{4}}$ then $n \ln \frac{1}{4} < \ln \varepsilon$ and $\left(\frac{1}{4}\right)^n < \varepsilon$

So $\left|\dfrac{x^{2n}}{1+x^{2n}}\right| < \dfrac{\left(\frac{1}{4}\right)^n}{1} < \varepsilon$

Scratchwork: want $\dfrac{x^{2n}}{1+x^{2n}} < \varepsilon$ since on $(0 \ \frac{1}{2})$

$\dfrac{x^{2n}}{1+x^{2n}} < \dfrac{x^{2n}}{1} < \left(\frac{1}{4}\right)^n$ it is sufficient to take $\left(\frac{1}{4}\right)^n < \varepsilon$

So $n > \dfrac{\ln \varepsilon}{\ln \frac{1}{4}}$