Hw 7 Solutions

Problem 1

a) $[3,4] \cup [5,6]$

b) $[1,2] \cup [5,6] \cup (11,12)$

c) $(0, \frac{1}{2}]$.

Problem 2

Suppose $f\ [a,b] \to \mathbb{R}$ is continuous

Let $S = f([a,b])$, first we will prove S is bounded below:

a) By contradiction assume S is not bounded below, then there is a sequence $\{x_n\}$ in $[a,b]$ s.t $f(x_n) < -n$

$\{x_n\}$ is bounded so it has a convergent subsequence $\{x_{n_k}\}$ $x_{n_k} \to x_0$ and $x_0 \in [a,b]$, since $[a,b]$ is closed; let $y_0 = f(x_0)$, then $y_0 \in S$ and $f(x_{n_k}) \to y_0 = f(x_0)$ because f is conctinuous, so $\{f(x_{n_k})\}$ is convergent and therefore it must be bounded, but $f(x_{n_k}) < -n_k \leq -k$, impossible

So S is bounded below, therefore it has an inf L.

Now we shall prove S is closed: let $\{y_n\}$ be a sequence in S converging to y_0, then $y_n = f(x_n)$ and $\{x_n\}$ is a sequence in $[a,b]$ so it is bounded and therefore it has a convergent subsequence $\{x_{n_k}\}$ that must converge to some $x_0 \in [a,b]$, therefore $f(x_{n_k}) \to f(x_0)$ because f is continuous, but $f(x_{n_k}) \to y_0$ so $y_0 = f(x_0)$ so y_0 is the image of $x_0 \in [a,b]$, therefore $y_0 \in S$

We could find a sequence $\{y_n\}$ in S $y_n \to l$. since $\forall n\ \exists y_n \in S$ $l - \frac{1}{n} < y_n < l + \frac{1}{n}$, since S is closed l must then be in S, so $l = \inf S$.

Problem 3

$f\ \mathbb{R} \to \mathbb{R}$

$f(x) = x$

Problem 4

a) F not guaranteed if f is not continuous

Ex $f\ [1,2] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } 2 \geq x > 1 \\ 0 & \text{if } x = 1 \end{cases}$$

b) T by the extreme value th

c) F, not guaranteed because $(1,2)$ is not closed

Ex $f\ (1,2) \to \mathbb{R}$

$f(x) = \frac{1}{x-1}$

d) F Ex $f\ [1,2] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 1/2 \\ 0 & \text{if } \frac{1}{2} < x \leq 2 \end{cases}$$

is bounded but not continuous

e) F Ex $f\ (1,2) \to \mathbb{R}$

$f(x) = x$

has no max

f) T (Th proved in class)

g) T (the image of a closed and bounded set under continuous function is closed, see proof of the extreme value th)

h) T (same as above)

i) F Ex $f\ [1,2]\cup[3,4] \to \mathbb{R}$

$f(x) = x$

Problem 5

Consider the function $f\ [0, \frac{\pi}{2}] \to \mathbb{R}$

$f(x) = x\sin x - \frac{1}{2}$

f is continuous

$f(0) = -\frac{1}{2} < 0$

$f(\frac{\pi}{2}) = \frac{\pi}{2} - \frac{1}{2} > 0$

by the intermediate value theorem ther must be $0 < x_0 < \frac{\pi}{2}$ s.t $f(x_0) = 0$

so $x_0 \sin x_0 - \frac{1}{2} = 0$ and therefore

$x_0 \sin x_0 = \frac{1}{2}$

Problem 6

a) $f : (0,1) \to \mathbb{R}$

$f(x) = \frac{1}{x}$ $\qquad f((0,1)) = (1, +\infty)$

b) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = \arctan(x)$

$f(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$

c) same as a)

d) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x$

Problem 7

a) yes. Given $\varepsilon > 0$ we need to find δ s.t. $|x^2 - y^2| < \varepsilon$ if $x, y \in (0,1)$, $|x-y| < \delta$

Given ε, take $\delta = \frac{\varepsilon}{2}$ then

$$|x^2 - y^2| = |x-y|\,|x+y| \leq \frac{\varepsilon}{2}(|x| + |y|) \leq$$

$$\leq \frac{\varepsilon}{2} \cdot 2 = \varepsilon \quad \text{if } 0 < x < 1 \text{ and } 0 < y < 1$$

Notice that this is not going to

work if we replace $(0,1)$ with $\mathbb{R}$ as Domain for f.

Alternatively if $\{x_n\} \in (0,1)$ and $\{y_n\} \in (0,1)$ and $(x_n - y_n) \to 0$ then

$f(x_n) - f(y_n) = x_n^2 - y_n^2 = (x_n - y_n)(x_n + y_n)$

and $\underbrace{-2|x_n - y_n|}_{\to 0} \le (x_n^2 - y_n^2) \le \underbrace{2|x_n - y_n|}_{\to 0}$

So by the squeeze theorem $x_n^2 - y_n^2 \to 0$

b) No take $u_n = n$ $\quad v_n = n + \frac{1}{n}$

$u_n - v_n \to 0$ but $f(u_n) - f(v_n) =$

$n^3 - (n^3 + 3n + 3\frac{1}{n} + \frac{1}{n^3}) \not\to 0$

Alternatively

take $\varepsilon = 1/2$ $\forall \delta$ we can find $x, y \in (0, +\infty)$

$|x - y| < \delta$ but $|x^3 - y^3| = |x - y||x^2 + y^2 + xy| > 1/2$

Take $x > \frac{1}{\sqrt{\delta}}$ $\quad y = x + \frac{\delta}{2}$ then

$|x-y||x^2 + y^2 + xy| \geq \frac{\delta}{2}(x^2 + x^2 + x\delta + \frac{\delta^2}{4} + x^2 + x\frac{\delta}{2}) >$

$> \frac{\delta}{2}(\frac{1}{\delta} + \frac{1}{\delta} + \text{other positive stuff}) > \frac{1}{2}$

c) yes given ε take $\delta = \frac{\varepsilon}{2}$ then

if $|x-y|<\delta$ $|f(x)-f(y)| = |(2x+1)-(2y+1)| =$

$2|x-y| < 2\frac{\varepsilon}{2} = \varepsilon$

Alternatively

if $\{u_n\}, \{v_n\}$ are sequences in $(0,+\infty)$

and $u_n - v_n \to 0$ then $f(u_n) - f(v_n) =$

$(2u_n+1) - (2v_n+1) = 2(u_n - v_n) \to 0$

Problem 8

Given ε take $\delta < \frac{\varepsilon}{M}$ then if

$|x-y|<\delta$ $|f(x)-f(y)| \leq M|x-y| < M\frac{\varepsilon}{M} = \varepsilon$