Homework 6

Solutions

Problem 1

a) We need to prove $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D - \{x_0\} \ |x - x_0| < \delta \Rightarrow |f(x) + g(x) - (A+B)| < \varepsilon$

Given $\varepsilon > 0$, we know there are d_1 and d_2 s.t if $|x - x_0| < d_1$ $x \neq x_0$ then $|f(x) - A| < \frac{\varepsilon}{2}$ and if $|x - x_0| < d_2$ $x \neq x_0$, then $|g(x) - B| < \frac{\varepsilon}{2}$

take $d = \min(d_1, d_2)$ then if $|x - x_0| < d$, $x \neq x_0$ we have

$|f(x) + g(x) - A - B| \leq |f(x) - A| + |g(x) - B| \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$;

b) Assume $x_n \to x_0$ then $f(x_n) \to A$ and $g(x_n) \to B$, therefore by the product rule for limits of sequences $f(x_n) \cdot g(x_n) \to AB$

Problem 2

a) We need to prove that $\forall x_0 \in \mathbb{R}$ $\lim\limits_{x \to x_0} x = x_0$: clearly if $x_n \to x_0$ then $f(x_n) = x_n \to x_0$

b) We need to prove that $\forall x_0 \in \mathbb{R}$ $\lim\limits_{x \to x_0} ax + b = ax_0 + b$

if $x_n \to x_0$ then $ax_n \to ax_0$ and $ax_n + b \to ax_0 + b$ by limit rules for sequences

c) We need to prove that $\forall x_0 \in \mathbb{R}$ $\lim\limits_{x \to x_0} x^n = x_0^n$:

if $x_k \to x_0$ then $\underbrace{x_k \cdot x_k \cdot \cdots \cdot x_k}_{n \text{ times}} \to x_0^n$

by limit rules for sequences (to be precise, we can use induction on n to prove that if we have n sequences $x_k^1 \to A_1, x_k^2 \to A_2, \dots, x_k^n \to A_n$, then their product $x_k^1 \cdot x_k^2 \cdots x_k^n \to A_1 A_2 \cdots A_n$ but a formal proof is not necessary to get full credit)

e) We need to prove that $\forall x_0 \in \mathbb{R}$ $x_0 \neq 0$ $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$

if $x_n \to x_0 \neq 0$ then $\frac{1}{x_n} \to \frac{1}{x_0}$

by limit rules for sequences

Problem 3

a) False : if f is discontinuous and $g = -f$ then $f+g$ is the constant function equal to zero, which is continuous

b) True : if $x_0 \in D$, the domain of f and g and $\{x_n\}$ is a sequence in D s.t $x_n \to x_0$ then $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$, because f and g are continuous, then $f(x_n)+g(x_n) \to f(x_0)+g(x_0)$ by a limit theorem for sequences

c) True: if $x_0 \in D$, the domain of f and $\{x_n\}$ is a sequence in D s.t $x_n \to x_0$ then $f(x_n) \to f(x_0)$, because f is continuous and $f^2(x_n) = f(x_n) \cdot f(x_n) \to f(x_0)^2$ by a theorem for limits of sequences

d) False $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is not continuous at 0, but f^2 is the constant function equal to 1, which is continuous

e) True the composition of continuous functions is continuous: proof assume $x_0 \in D$, the domain of f and $\{x_n\}$ is a sequence in D s.t $x_n \to x_0$ then $f(x_n) \to f(x_0)$ because f is continuous and $g(f(x_n)) \to g(f(x_0))$ because g is continuous

f) False $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ 3 & \text{if } x > 0 \end{cases}$$

is not continuous at 0 but $\forall x$ $f(f(x)) = 3$ so $f \circ f$ is the constant function equal to 3, which is continuous

Problem 4

Given $\varepsilon > 0$ take $\delta = \varepsilon$ then if $|x| < \delta$ $|\sin x| < |x| < \varepsilon$

Therefore $\lim_{x \to 0} \sin x = 0 = \sin 0$, so $\sin x$ is continuous at 0.

$\lim_{x \to x_0} \sin x = \lim_{x \to x_0} \sin x - \sin x_0 + \sin x_0$, I want to

show $\lim_{x \to x_0} \sin x - \sin x_0 = 0$

Given $\varepsilon > 0$ take $\delta = \varepsilon$ then if $|x - x_0| < \delta$

$|\sin x - \sin x_0| = 2 \left| \sin \frac{x - x_0}{2} \right| \left| \cos \frac{x + x_0}{2} \right| \leq 2 \left| \sin \left(\frac{x - x_0}{2} \right) \right|$

$\leq 2 \cdot \frac{|x - x_0|}{2} \leq \varepsilon$

Therefore $\lim_{x \to x_0} \sin(x) - \sin(x_0) + \sin(x_0) =$

$0 + \sin x_0 = \sin x_0$ and $\sin x$

is continuous at x_0

Problem 5

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

both f and g are discontinuous at 0 but $f \cdot g$ is the constant function equal to 0, which is continuous everywhere

Problem 6

Is $\lim_{x \to 0} f(x) = 0$?

Suppose $x_n \to 0$ then $f(x_n) = x_n$ or 0 and therefore $f(x_n) \to 0$

Is $\lim_{x \to 1} f(x) = 1$? No we can find a sequence $\{x_n\}$ $x_n \to 1$ x_n all irrational

for example $x_n = 1 + \frac{\sqrt{2}}{n}$; (if x_n was rational then $n \cdot x_n - 1 = \sqrt{2}$ would be too)

So $x_n \to 1$ $f(x_n) \to 0$ but $0 \neq f(1) = 1$

Problem 7

Everywhere in its domain, because it is the quotient of continuous functions

Problem 8

$\sin x$ is continuous everywhere

$\frac{1}{x}$ is continuous on its domain $x \neq 0$

so $\sin(\frac{1}{x})$ is continuous everywhere in its domain $x \neq 0$