Hw 6

Read chapter 3 of the textbook.
Main skills:

- You need to know the definition of limit of a function.
- You need to know the definition of continuous function

Do the following problems:

1. Let $f : D \to R$ and $g : D \to R$ be two functions, x_0 be a limit point for D and $\lim_{x\to x_0} f(x) = A, \quad \lim_{x\to x_0} g(x) = B$.
   - Prove that $\lim_{x\to x_0}(f(x) + g(x)) = A + B$ using the ϵ, δ definition of limit.
   - Prove that $\lim_{x\to x_0}(f(x)g(x)) = AB$ any way you want.
2. Prove that the following functions from their natural domain to R are continuous:
   - $f(x) = x$
   - $g(x) = ax + b$
   - $h(x) = x^n$, $n \in N$
   - $s(x) = \frac{1}{x}$
3. For each of the following statements say whether it is True or False: give a proof or a conterexample to justify your answer.
   - If the function $f + g$ is continuous, then f and g are continuous functions.
   - If the functions f and g are continuous, then $f + g$ is a continuous function.
   - If f is a continuous function then f^2 (defined as $f^2(x) = f(x)^2$) is continuous
   - If f^2 is continuous then f is continuous.
   - If f is a continuous function then $f \circ f$ (the composition of f with itself) is continuous.
   - If $f \circ f$ is continuous then f is continuous.
4. Use the inequality :$|\sin x| \leq |x|$ (you do not nered to prove this inequality) to prove that $\lim_{x\to 0} \sin x = 0$. Use this limit and the trigonometric identity: $\sin x - \sin x_0 = 2 \sin \frac{x-x_0}{2} \cos \frac{x+x_0}{2}$ to prove that $\lim_{x\to x_0} \sin x = \sin x_0$ and therefore the sin function is continuous.
5. Find an example of functions $f : R \to R$ and $g : R \to R$ such that f is discountinuous at 0 but fg is continuous at 0. Can fg be continuous at 0 if both f and g are discontinuous at 0 ?

6. Consider the function $f : R \to R$ defined by:

$$f(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

Is f continuous at 0 ?
Is f continuous at 1 ?

7. Consider the function $f : R - \{1\} \to R$ defined by: $f(x) = \frac{x^2-1}{x-1}$ where is f continuous ?

8. Consider the function $f : R - \{0\} \to R$ defined by: $f(x) = \sin \frac{1}{x}$ where is f continuous ?