Hw 5 Solutions

Problem 1

a) Given that $\ln 1 = 0$ and $\ln x$ is monotone increasing $\lim_{l \to +\infty} \ln(l) \neq 0$
so the series diverges

b) $\lim_{l \to +\infty} \frac{\frac{l+1}{3^{l+1}}}{\frac{l}{3^l}} = \lim_{l \to +\infty} \frac{l+1}{l} \cdot \frac{3^l}{3^{l+1}} = \frac{l(1+\frac{1}{l})}{l} \cdot \frac{1}{3} = \frac{1}{3}$ so by the ratio test the series converges

Problem 2

a) By the alternating series criterion this series converges,

since $\lim_{l \to \infty} \frac{1}{7l-5} = \frac{1}{l} \cdot \frac{1}{7-\frac{5}{l}} = 0 \cdot \frac{1}{7} = 0$ and $\frac{1}{7(l+1)-5} \leq \frac{1}{7l-5}$),

but it is not absolutely convergent (see problem 1 a) in previous hw)

b) $\left| \frac{\sin(l)}{l^2+2} \right| < \frac{1}{l^2}$ and $\sum_{l=1}^{\infty} \frac{1}{l^2}$ converges, so by comparison test the series is absolutely convergent

c) $\lim_{l \to +\infty} (-1)^l \frac{l^2}{l(l+1)} = (-1)^l \frac{l^2}{l^2(1+\frac{1}{l})} = (-1)^l \cdot \frac{1}{1+\frac{1}{l}} = \text{DNE}$ since if $l = 2k$ we get the subsequence $\frac{1}{1+\frac{1}{2k}} \to 1$ if $l = 2k-1$ we get the subsequence

$-1 \cdot \frac{1}{1+\frac{1}{2k-1}} = \frac{-1}{1+\frac{1}{k} \cdot \frac{1}{2-\frac{1}{k}}} = -1$ so in particular $\lim_{l \to +\infty} (-1)^l \frac{l^2}{l(l+1)} \neq 0$

so the series diverges.

d) converges by the alternating series test, since

$\frac{1}{\sqrt{l+1+1}} < \frac{1}{\sqrt{l+1}}$ and $\frac{1}{\sqrt{l+1}} \to 0$ (no points off in hw for not proving it, but here is a proof:

given $\varepsilon > 0$ take $N > \frac{1}{\varepsilon^2} - 1$ then if $n \geq N$ $n > \frac{1}{\varepsilon^2} - 1$ so $n+1 > \frac{1}{\varepsilon^2}$

So $\varepsilon^2 > \frac{1}{n+1}$ and $\varepsilon > \frac{1}{\sqrt{n+1}}$)

but not absolutely since $\frac{1}{\sqrt{l+1}} > \frac{1}{\sqrt{l+l}} = \frac{1}{\sqrt{2}\sqrt{l}}$ and $\sum_{l=1} \frac{1}{l^{1/2}}$ diverges

Problem 3

Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both absolutely convergent

Let $s_n = \sum_{k=1}^{n} |a_k|$, $\sigma_n = \sum_{k=1}^{n} |b_k|$ then $\{s_n\}$ and $\{\sigma_n\}$

both converge. We need to show $\{t_n\}$ converges

if $t_n = \sum_{k=1}^{n} |a_k + b_k|$

t_n is clearly increasing and

$0 \le t_n \le s_n + \sigma_n$ since for each k $|a_k + b_k| \le |a_k| + |b_k|$

s_n and σ_n are bounded, because they are convergent

so t_n is bounded, and an increasing bounded sequence converges

Problem 4

Using the ratio test $\lim_{n\to+\infty} \dfrac{\frac{|x|^{n+1}}{3(n+1)}}{\frac{|x|^n}{3n}} = \lim_{n\to+\infty} |x| \dfrac{n}{n+1} = |x|$

so if $|x| > 1$ the series diverges, if $|x| < 1$ the series

converges absolutely if $|x| = 1$ we do not know, so we

need to consider separately the cases $x = 1$ and $x = -1$

If $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{3n}$ which diverges

if $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$ which converges by the

alternating series test, but not absolutely.

So the series converges for $-1 \le x < 1$

it converges absolutely for $-1 < x < 1$

Problem 5

Let $s_n = \sum_{i=1}^{n} b_i$

$$s_{6n} = a_1+a_2+a_3-a_4-a_5-a_6+\cdots = \overbrace{(a_1+a_2+a_3-a_4-a_5-a_6)+\cdots+(a_{6n-11}+a_{6n-10}+a_{6n-9}-a_{6n-8}-a_{6n-7}-a_{6n-6})}^{s_{6(n-1)}}+$$

$$+\overbrace{(a_{6n-5}+a_{6n-4}+a_{6n-3}-a_{6n-2}-a_{6n-1}-a_{6n})}^{\geq 0}, \text{ equivalently}$$

$$s_{6n}=a_1+a_2+a_3-(a_4-a_7)-(a_5-a_8)-(a_6-a_9)\cdots-(a_{6n-6}-a_{6n-3})-a_{6n-2}-a_{6n-1}-a_{6n}$$

so $s_{6n} \geq 0$ and s_n is increasing and bounded above by $a_1+a_2+a_3$

so s_{6n} converges to some $\ell \in \mathbb{R}$

$s_{6n+1} = s_{6n}+a_{6n+1}$ also converges to ℓ, since $a_{6n+1} \to 0$

$s_{6n+2} = s_{6n}+a_{6n+1}+a_{6n+2}$ also converges to ℓ ...

Clearly for any k, $0 \leq k \leq 5$ $\lim_{n\to+\infty} s_{6n+k} = \ell$

Therefore $\forall \varepsilon\ \forall k \in \{0,1,2,3,4,5\}\ \exists N_k\ \forall n \geq N_k\ |s_{6n+k}-\ell| < \varepsilon$ and so

given ε if $N > \max\{6N_0, 6N_1+1, 6N_2+2, 6N_3+3, 6N_4+4, 6N_5+5\}$

$|s_n-\ell| < \varepsilon$ so $\lim_{n\to+\infty} s_n = \ell$