Homework 4
Solutions

Problem 1

a) $\frac{1}{7l-5} > \frac{1}{7l}$. $\sum_{l=1}^{\infty} \frac{1}{l}$ diverges and $\sum_{l=1}^{\infty}\frac{1}{7l} = \sum_{l=1}^{\infty}\frac{1}{7}\frac{1}{l}$ so $\sum_{l=1}^{\infty}\frac{1}{7l}$ diverges also (by problem 3)

so $\sum_{l=1}^{\infty} \frac{1}{7l-5}$ diverges by comparison with $\sum_{l=1}^{\infty}\frac{1}{7l}$

b) $\frac{1}{3l+5} > \frac{1}{3l+l}$ if $l \geq 5$. Again $\sum_{l=1}^{\infty}\frac{1}{4l} = \sum_{l=1}^{\infty}\frac{1}{4}\frac{1}{l}$ so $\sum_{l=1}^{\infty}\frac{1}{4l}$ diverges and $\sum_{l=1}^{\infty}\frac{1}{3l+5}$ diverges also by comparison with $\sum_{l=1}^{\infty}\frac{1}{4l}$

you could also here calculated $\lim_{l\to+\infty}\frac{\frac{1}{3l+5}}{\frac{1}{l}} = \lim_{l\to+\infty}\frac{l}{3l+5} = \lim \frac{l}{l(3+\frac{5}{l})} = \frac{1}{3}$ to justify divergence using limit test

c) $\frac{1}{l^2+2} < \frac{1}{l^2}$ so the series converges by comparison with $\sum_{l=1}^{\infty}\frac{1}{l^2}$

d) $\frac{1}{l^2+l} < \frac{1}{l^2}$ so again the series converges by comparison with $\sum_{l=1}^{\infty}\frac{1}{l^2}$

e) $\sum_{l=1}^{\infty}\frac{5}{3^l} = \sum_{l=1}^{\infty} 5\left(\frac{1}{3}\right)^l$ $\quad \frac{1}{3}<1$ so $\sum_{l=1}^{\infty}\left(\frac{1}{3}\right)^l$ converges and therefore $\sum_{l=1}^{\infty}\frac{5}{3^l}$ converges (by a theorem proved in class)

f) $5\cdot 3^l > 3^l$ and $\sum_{l=1}^{\infty} 3^l$ diverges. so $\sum_{l=1}^{\infty} 5\cdot 3^l$ diverges by problem 3

g) $\frac{2}{3^k+5} < \frac{2}{3^k}$ and $\sum_{k=1}^{\infty} \frac{2}{3^k} = \sum_{k=1} 2 \cdot \frac{1}{3^k}$ so the series converges

h) $\lim_{k \to +\infty} \frac{\frac{2}{3^k-5}}{\frac{1}{3^k}} = \lim_{k \to +\infty} \frac{2 \cdot 3^k}{3^k-5} = \lim_{k \to +\infty} 2 \, \frac{\cancel{3^k}}{\cancel{3^k}} \, \frac{1}{\left(1-\frac{5}{3^k}\right)} = 2$

therefore the series behaves like $\sum_{k=1}^{\infty} \frac{1}{3^k}$ which converges

i) $\lim_{k \to +\infty} \frac{\frac{(k+2)^2}{(k^2+3)^2}}{\frac{1}{k^2}} = \lim_{k \to +\infty} \frac{k^2 \left(1+\frac{2}{k}\right)^2 \cdot k^2}{k^4 \left(1+3/k^2\right)^2} = 1$

so the series behaves like $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges.

Problem 2

Let $s_n = \sum_{i=0}^{n} a_i$ and $t_n = \sum_{i=0}^{n} b_i$

then $s_n + t_n = \sum_{i=0}^{n} (a_i + b_i)$

and $\lim_{n\to+\infty} s_n = a$ $\lim_{n\to+\infty} b_n = b$ so by the sum limit theorem for convergent sequences $\lim_{n\to+\infty} (s_n + t_n) =$

$= a + b$, and this means

$\sum_{i=0}^{\infty} (a_i + b_i) = a + b$

Problem 3

Let $s_n = \sum_{i=1}^{n} a_i$, and $t_n = \sum_{i=1}^{n} \alpha a_i$ $(\alpha \neq 0)$

assume by contradiction that

$\sum_{i=1}^{\infty} a_i$ diverges but $\sum_{i=1}^{\infty} \alpha a_i$ converges

then by a theorem proved in class

$\sum_{i=1}^{\infty} \frac{1}{\alpha} (\alpha a_i) = \sum_{i=1}^{\infty} a_i$ converges,

contradicting our original assumption

Note α could be < 0 so using
comperison th or limit th is problematic