Homework 3 Solutions

Problem 1: We need to prove $\forall K \in \mathbb{N}\ \exists M \in \mathbb{N}\ \forall n \geq M \quad a_n + b_n > K$

Given K we know there is $M_1 \in \mathbb{N}$ s.t $\forall n \geq M_1 \quad a_n > \frac{K}{2}$

and there is $M_2 \in \mathbb{N}$ s.t $\forall n \geq M_2 \quad b_n > \frac{K}{2}$; let $M = \max(M_1, M_2)$

then $\forall n \geq M \quad a_n + b_n > K$

Problem 2

a) We want to show $\lim_{n \to +\infty} \frac{n^2}{1+n} = +\infty$

Given K take $M > 2K$ then if $n \geq M \quad \frac{n^2}{1+n} \geq \frac{n^2}{n+n} = \frac{n}{2} > \frac{2K}{2} = K$

b) We want to show $\lim_{n \to +\infty} b_n = +\infty$

Given K take $M > K$ then if $n \geq M \quad \sum_{i=1}^{n} = 1+2+\cdots+n \geq n > K$

Problem 3

a) $a_n = \frac{1}{n^2}$, $b_n = n$

b) $a_n = \frac{1}{n}$, $b_n = n^2$

c) $a_n = \frac{1}{n}$, $b_n = n$

d) $a_n = \frac{(-1)^n}{n}$, $b_n = n$

Problem 4

a) Closed and open, not bounded so it is not sequentially compact

b) Closed, bounded, sequentially compact

c) Closed not bounded not sequentially compact

d) Open, not sequentially compact

e) Closed, not bounded not sequentially compact

f) Open, not sequentially compact

g) Neither, not sequentially compact

h) Closed, bounded, sequentially compact

i) Open not sequentially compact

j) Closed, bounded, sequentially compact

k) $S = \{0\}$ closed, sequentially compact

Problem 5

a) Monotonically decreasing, since $\frac{1}{n} \geq \frac{1}{n+1}$

bounded since $\left|\frac{1}{n}\right| \leq 1$

b) monotonically increasing, not bounded

c) Bounded

Problem 6

$\Rightarrow$: We know a convergent sequence is bounded.

$\Leftarrow$ Assume $\{a_n\}$ is monotonically decreasing and bounded, we would like to show:

$$\lim_{n \to +\infty} a_n = \inf\{a_n \mid n \in \mathbb{N}\} = L$$

Given $\varepsilon > 0$ by definition of inf

there is a_M s.t $L + \varepsilon > a_M$ and

$\forall n \geq M$ $L + \varepsilon > a_M \geq a_n$

since ($\{a_n\}$ is increasing)

also $\forall n \in \mathbb{N}$ $L - \varepsilon < a_n$, therefore

$\forall n \geq M$ $|a_n - L| < \varepsilon$

Therefore $\lim_{n \to +\infty} a_n = L$

Problem 7

a) The subsequence $\{\cos(\pi 2k)\}$ is constent equal to 1, therefore converges to 1. The subsequence $\{\cos(\pi(2k+1))\}$ is constent equal to -1, so it converges to -1, therefore $\lim_{n \to +\infty} \cos(\pi n)$ DNE

b) First we will proce the sepuence is bounded by proving $\forall n \in \mathbb{N}\ b_n \le 2$

Proof by induction

1) Base case : $b_1 = \sqrt{2} \le 2$ is true

2) Induction step : assume $b_n \le 2$, then we need to show $b_{n+1} = \sqrt{2 + b_n} \le 2$

$$\sqrt{2 + b_n} \le 2 \iff$$

$$2 + b_n \le 4 \iff$$

$$b_n \le 2$$ which is true by our induction assumption.

Now we will show $\{b_n\}$ is increasing :

$$b_{n+1} \geq b_n \iff$$

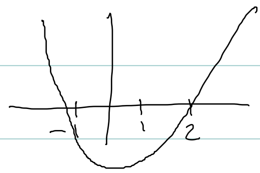
$$\sqrt{2+b_n} \geq b_n \iff$$

$$2 + b_n \geq b_n^2$$

Consider the parabola $y = x^2 - x - 2$

it has x intercepts where $x^2 - x - 2 = 0$

that is $x = 2$, $x = -1$ and it is looking up



so we can say

$\forall x \quad 0 \le x \le 2 \qquad x^2 - x - 2 \le 0$

or $2 + x \ge x^2$, since $b_n \le 2$

it is true $2 + b_n \ge b_n^2$

So our sequence is increasing and bounded. It must converge

c) We want to prove $\lim_{n \to +\infty} c_n = +\infty$ that is

$\forall K \in \mathbb{N} \; \exists M \in \mathbb{N} \; \forall n \ge M \quad a_n > K$

First we will prove by induction that $\forall n \in \mathbb{N}\ c_n \geq n$

1) Base case: obvious

2) Induction step: assume $c_n \geq n$

Then $c_{n+1} = 1 + 2c_n \geq 1 + 2n \geq n+1$

Therefore given $K \in \mathbb{N}$ take $M > K$

$\forall n \geq M \quad c_n > K$

d) $\{d_{2K}\}_{K\in\mathbb{N}}$ converges to 1 since

$$d_{2K} = \frac{2K}{2K+1} = \left(1 - \frac{1}{2K+1}\right)$$

$\{d_{2K-1}\}_{K\in\mathbb{N}}$ converges to -1 since

$$d_{2K-1} = -\frac{K}{K+1} = -\left(1 - \frac{1}{K+1}\right)$$

Therefore $\lim\limits_{n\to+\infty} d_n$ DNE

Problem 8

Assume $\lim_{n \to +\infty} x_n = +\infty$ and consider

the subsequence $\{x_{n_k}\}$

We need to prove $\forall h \in N \; \exists L \in N \; \forall k \geq L$

$a_{n_k} > h$

Given $h \in N \; \exists M \in N \quad \forall n \geq M \quad x_n > h$

So take L s.t $n_L \geq M$ then

$\forall k \geq L \quad n_k \geq M$ so $x_{n_k} \geq h$