Hw 3

Read chapter 2 of the textbook.

Main skills:

- You need to know the definition of limit of a sequence being ∞
- You need to know the definition of monotone and bounded sequence and relevant theorems.
- You need to know the definition of subsequence of a sequence and relevant theorems
- You need to know the definition of open and closed set and sequentially compact set in R.
- You need to know the definition of the number e .

Do the following problems:

1. Prove that if $\{a_n\}$ and $\{b_n\}$ are sequences and $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty}(a_n + b_n) = \infty$
2. Find the limit of the following sequences and give a proof:
   (a) $\{a_n\}$, where $a_n = \frac{n^2}{1+n}$
   (b) $\{b_n\}$, where $b_n = \sum_{i=1}^{n} i$
3. Find examples of sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to 0$ and $b_n \to \infty$ and :
   (a) $a_n \cdot b_n \to 0$
   (b) $a_n \cdot b_n \to \infty$
   (c) $a_n \cdot b_n \to 1$
   (d) $\lim_{n\to\infty}(a_n \cdot b_n)$ DNE
4. For each of the following subsets of R say if it is open or closed (or both or neither) and if it is sequentially compact
   (a) R
   (b) $\{1, 2, 3\}$
   (c) Z
   (d) $(-\infty, 1)$
   (e) $(-\infty, 1]$
   (f) $(-2, 1)$
   (g) $(-2, 1]$
   (h) $[-2, 1]$

(i) $(-2, 1) \cup (3, 5)$

(j) $[-2, 1] \cup [3, 5]$

(k) $\bigcap_{i=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$

No proofs necessary

5. For each of the following sequences say if it is monotonically increasing , monotonicaly decreasing, bounded:

   (a) $\{a_n\}$, where $a_n = \frac{1}{n}$

   (b) $\{a_n\}$, where $a_n = n$

   (c) $\{a_n\}$, where $a_n = \sin n$

6. Prove that a monotonically decreasing sequence converges if and only if it is bounded.

7. For each of the following sequences say whether it is convergent, it has limit ∞ or $-\infty$ or it has no limit, and justify your answer:

   (a) $\{a_n\}$, where $a_n = \cos(\pi\, n)$

   (b) $\{b_n\}$, where b_n is defined by :
   $b_1 = \sqrt{2}$
   $b_{n+1} = \sqrt{2 + b_n}$

   (c) $\{c_n\}$, where c_n is defined by :
   $c_1 = 1$
   $c_{n+1} = 1 + 2c_n$

   (d) $\{d_n\}$, where d_n is the following sequence :
   $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}, \cdots$

8. Prove that if $\lim_{n\to\infty} a_n = \infty$ and $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ then $\lim_{k\to\infty} a_{n_k} = \infty$