Homework 2
Solutions

Problem 1: proof by induction on n.

Base case: if $n=1$ the inequality becomes $1+x \geq 1+x$, obviously true

Induction step: assume $(1+x)^n \geq 1+nx$ for some $n \in \mathbb{N}$

then multiplying both sides by $1+x$ we have

$(1+x)^n(1+x) \geq (1+nx)(1+x)$ (note that $1+x>0$ since $x>-1$)

therefore $(1+x)^{n+1} \geq 1+(n+1)x+nx^2 \geq 1+(n+1)x$. (since $nx^2 \geq 0$).

Problem 2

a) $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

b) $b_4 = 3+4+5+6 = 18$

c) $c_4 = \sum_{i=1}^{4}(4+2) = 6+6+6+6 = 24$

Problem 3

Assume $\lim_{n\to+\infty} a_n = a$, if $c=0$ then $\{ca_n\}$ is the constant sequence equal to 0, so $\lim_{n\to+\infty} ca_n = 0 = c\cdot a$

If $c \neq 0$, we need to prove:

$\forall \varepsilon>0\ \exists M\in\mathbb{N}\ \forall n\geq M\quad |ca_n - ca| < \varepsilon$:

given ε, choose M s.t. $\forall n \geq M$ $|a_n - a| < \frac{\varepsilon}{|c|}$, then

$|c|\cdot|a_n - a| = |ca_n - ca| < \varepsilon$ for all $n \geq M$

Problem 4

a) We want to prove $\{a_n\}$ converges to $c=1$, that is we need to prove:

$\forall \varepsilon > 0 \ \exists M \in \mathbb{N} \ \forall n \geq M \ |a_n - 1| < \varepsilon$:

given $\varepsilon > 0$ take $M > \frac{1}{\varepsilon} - 2$ then if $n \geq M$ $n+2 \geq M+2 > \frac{1}{\varepsilon}$

and $\left| \frac{n+1}{n+2} - 1 \right| = \left| \frac{n+1-n-2}{n+2} \right| = \frac{1}{n+2} \leq \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$

How did I find M to start with?

by solving $\left| \frac{n+1}{n+2} - 1 \right| < \varepsilon$:

$\left| \frac{n+1}{n+2} - 1 \right| < \varepsilon \iff$

$\left| \frac{n+1-n-2}{n+2} \right| < \varepsilon \iff$

$\frac{1}{n+2} < \varepsilon \iff$

$n+2 > \frac{1}{\varepsilon} \iff$

$n > \frac{1}{\varepsilon} - 2$ but we do not usually include this scratchwork in our final proof

b) We want to prove $\{b_n\}$ converges to 0 :

since $0 \leq \frac{1+(-1)^n}{n} \leq \frac{2}{n}$, by the squeeze theorem, it is sufficient to prove $\lim_{n \to +\infty} \frac{2}{n} = 0$ that is we need to prove

$\forall \varepsilon > 0 \ \exists M \in \mathbb{N} \ \forall n \geq M \ \left|\frac{2}{n}\right| < \varepsilon$

Given ε, take $M > \frac{2}{\varepsilon}$, then if $n \geq M$ $\left|\frac{2}{n}\right| = \frac{2}{n} < \frac{2}{\frac{2}{\varepsilon}} = \varepsilon$

or it would be OK to say that we proved in class $\lim_{n \to +\infty} \frac{1}{n} = 0$ so $\lim_{n \to +\infty} 2 \cdot \frac{1}{n} = 2 \cdot 0 = 0$

c) We want to prove $\{c_n\}$ converges to 0

In class we proved $\lim_{n \to +\infty} \frac{1}{n} = 0$, by one of the limit laws we then have $\lim_{n \to +\infty} (-1) \cdot \frac{1}{n} = 0$ and by the squeeze theorem, since

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

we have that

$$\lim_{n \to +\infty} \frac{\sin(n)}{n} = 0$$

d) $\lim_{n \to +\infty} \frac{2}{n} = 0$ (proved in b) above)

$\lim_{n \to +\infty} \left(\frac{1}{2}\right)^n = 0$ since $\frac{1}{2} < 1$ and we proved in class

that if $0 < a < 1$ $\lim_{n \to +\infty} a^n = 0$

therefore $\lim_{n \to +\infty} \frac{2}{n} + \left(\frac{1}{2}\right)^n = 0$

or we can prove directly $\lim_{n\to+\infty}\left(\frac{1}{2}\right)^n = \lim_{n\to+\infty}\frac{1}{2^n} = 0$

We need to prove

$\forall \varepsilon > 0 \ \exists M \in \mathbb{N} \ \forall n \geq M \quad \left|\frac{1}{2^n}\right| < \varepsilon$

given ε take $M > \frac{\ln(1/\varepsilon)}{\ln 2}$ then if

$n \geq M \qquad n > \frac{\ln(1/\varepsilon)}{\ln 2}$ so $n \ln(2) > \ln\left(\frac{1}{\varepsilon}\right)$ so

$\ln(2^n) > \ln\left(\frac{1}{\varepsilon}\right)$ so $2^n > \frac{1}{\varepsilon}$ so $\varepsilon > \frac{1}{2^n} = \left|\frac{1}{2^n}\right|$

e) $\lim_{n\to+\infty}\frac{1}{n} = 0$ (proved in class) therefore

$\lim_{n\to+\infty}\frac{1}{n}\cdot\frac{1}{n} = 0$ and $\lim_{n\to+\infty}(-1)\frac{1}{n}\cdot\frac{1}{n} = 0$

since $-\frac{1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$, by the squeeze

theorem $\lim_{n\to+\infty}\frac{\sin}{n^2} = 0$ and $\lim_{n\to+\infty} 2\cdot\frac{\sin(n)}{n^2} = 0$

$\lim_{n\to+\infty}\frac{n+1}{n+2} = 1$ (proved in part b) therefore

$\lim_{n\to+\infty}\frac{2\sin(n)}{n^2}\cdot\frac{n+1}{n+2} = 0$; $\lim_{n\to+\infty} 3 = 3$, so finally

$\lim_{n\to+\infty} 2\frac{\sin(n)}{n^2}\frac{n+1}{n+2} + 3 = 3$

Problem 5

Assume by contradiction that $a > b$ and take $0 < \varepsilon < \frac{a-b}{2}$

We can find M_1 and M_2 s.t $\forall n \geq M_1 \quad |a_n - a| < \varepsilon$

and $\forall n \geq M_2 \quad |b_n - b| < \varepsilon$, therefore

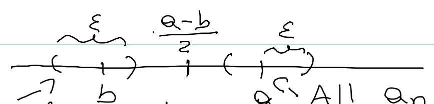
$\forall n \geq \max\{M_1, M_2\} \quad |a_n - a| < \frac{\varepsilon}{2}$ and $|b_n - b| < \varepsilon$

Choose any $n > \max\{M_1, M_2\}$, then we have

$$b_n < b + \varepsilon < b + \frac{a-b}{2} = \frac{a+b}{2}$$

$$a_n > a - \varepsilon > a - \frac{a-b}{2} = \frac{a+b}{2}$$

so $a_n > b_n$, contradiction



All b_n for $n > M_2$ are here

All a_n for $n \geq M_1$ are here