Homework 1 solutions

Problem 1

a) S is bounded above, below, $\sup S = 3$ This is a max

$\inf S = 1$ this is a min

b) S is not bounded above. It is bounded below. $\inf S = 2$

This is a min

c) S is not bounded above or below

d) S is bounded above and below. $\sup S = 2$ this is a

max. Inf S 1 $1 \notin S$ so it is not a min

e) S is bounded above and below. $\sup S = 1$, not a max

$\inf S = \frac{2}{3}$ [the sequence $a_n = \frac{n+1}{n+2}$ is increasing,

since $\frac{n+1}{n+2} < \frac{(n+1)+1}{(n+1)+1}$], this is a min

Problem 2

(a) Let $m = \min(\inf A, \inf B)$, $l_1 = \inf(A)$, $l_2 = \inf B$; we shall prove

1) m is a lower bound for $A \cup B$: if $x \in A \cup B$ then either $x \in A$, therefore

$x \geq l_1 \geq m$ or $x \in B$ and $x \geq l_2 \geq m$, so in any case if

$x \in A \cup B$ $x \geq m$ so m is a lower bound for $A \cup B$

2) Given any positive ε, $m + \varepsilon$ is not a lower bound for $A \cup B$:

if $m = l_1$ then there is $x \in A$ and therefore $x \in A \cup B$ s.t

$m + \varepsilon = l_1 + \varepsilon > x$. Similarly if $m = l_2$ there is $x \in B$ and therefore $x \in A \cup B$ s.t. $m + \varepsilon = c_2 + \varepsilon > x$

(b) Let $m = \max(\inf A, \inf B)$, we shall show m is a lower bound for $A \cap B$ therefore $m \leq \inf(A \cap B)$:

Given $x \in A \cap B$ then $x \in A$ therefore $x \geq \inf(A)$ and $x \in B$ therefore $x \geq \inf(B)$. so $x \geq m$

c) We shall show $\inf T$ is a lower bound for S: if $x \in S$ then $x \in T$ so $\inf T \leq x$

Problem 3

$A = \{1, 4\}$ $\quad \inf A = 1$

$B = \{3, 4\}$ $\quad \inf B = 3$

$A \cap B = \{4\}$ $\quad \inf(A \cap B) = 4$

$\max(1, 3) = 3 < \inf(A \cap B) = 4$

Problem 4

$\frac{n}{n+1} \leq 1$ so 1 is an upperbound for S.

Given $\varepsilon > 0$ we need to show $1 - \varepsilon$ is not an upper bound for S, that is there is $n \in \mathbb{N}$ s.t $\frac{n}{n+1} > 1 - \varepsilon$

Solving for n we get

$\frac{n}{n+1} > 1-\varepsilon \iff$

$n > n - n\varepsilon + 1 - \varepsilon \iff$

$n\varepsilon > 1-\varepsilon \iff$

$n > \frac{1-\varepsilon}{\varepsilon}$

Therefore if $n > \frac{1-\varepsilon}{\varepsilon}$ $\frac{n}{n+1} > 1-\varepsilon$ and $1-\varepsilon$ is not an upper bound for S

$\frac{1}{2}$ is a lower bound for S since

$\frac{n}{n+1} \geq \frac{1}{2}$ is equivelent to

$2n \geq n+1$ which is equivalent to

$n \geq 1$ which is true for all $n \in \mathbb{N}$

$\frac{1}{2}$ is a min for S since when $n=1$ we get $\frac{1}{1+1} = \frac{1}{2}$

so $\frac{1}{2} = \inf S$

Problem 5

$S = \left\{ \frac{1}{n} + (-1)^n \right\}$

$L = -1$. Proof

If n is even $-1 \leq \frac{1}{n} + 1$ is true

If n is odd $-1 \leq \frac{1}{n} - 1$ is true

so $\forall x \in S$ $-1 \leq x$

Now we need to prove $\forall \varepsilon > 0 \ \exists x \in S \ \ -1+\varepsilon > x$

Let's look for x of the form $\frac{1}{n} - 1$, with n odd

Scratchwork

$-1+\varepsilon > \frac{1}{n} - 1 \iff$

$n > \frac{1}{\varepsilon}$ End

So given $\varepsilon > 0$ take n odd s.t $n > \frac{1}{\varepsilon}$, then

$\varepsilon > \frac{1}{n}$ and $-1+\varepsilon > -1 + \frac{1}{n} \in S$

$s = \frac{3}{2}$. Proof:

if n is odd we need to check $\frac{1}{n} - 1 \leq \frac{3}{2} \iff$

$\frac{1}{n} \leq \frac{5}{2} \iff$

$2 \leq 5n$ True

If n is even we need to check $\frac{1}{n} + 1 \leq \frac{3}{2} \iff$

$\frac{1}{n} \leq \frac{1}{2} \iff$

$2 \leq n$ True

Therefore $\frac{3}{2}$ is an upper bound for S

and $\frac{3}{2} \in S$ since when $n=2$ $\frac{1}{n} + (-1)^n = \frac{3}{2}$

so $\frac{3}{2} = \max(S) = \sup(S)$