Determinant of a general nxn matrix $M = (a_{ij})$

If n=1 detM= a_{11}.

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

M_{ij} is the matrix I obtain from M by deleting row i and column j.

Expansion along ith row

$det(M) = (-1)^{i+1} a_{i1} det(M_{i1}) + (-1)^{i+2} a_{i2} det(M_{i2}) + \cdots + (-1)^{i+n} a_{in} det(M_{in})$

Expansion along jth column

$det(M) = (-1)^{1+j} a_{1j} det(M_{1j}) + (-1)^{2+j} a_{2j} det(M_{2j}) + \cdots + (-1)^{n+j} a_{nj} det(M_{nj})$

$C_{ij} = (-1)^{i+j} det(M_{ij})$ is caled the cofactor of a_{ij}

$det(M_{ij})$ is called the minor of a_{ij}

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$det(M) =$$

$$a_{11} \left| \begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix} \right| - a_{12} \left| \begin{matrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{matrix} \right| + a_{13} \left| \begin{matrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \right|$$

Compute the determinant of $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$det(M) = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$$

$$= (45 - 48) - 2(36 - 42) + 3(32 - 35) = 0$$

Fact: $det(A) = 0$ if and only if A is singular

Determinant of a triangular matrix

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$det(M) = a_{11}det(M_{11}) = a_{11}a_{22}det(M_{11}') + \cdots$$

$$= 1 * 5 * 8 * 10$$

Determinant and elementary operations.

- If B is obtained from A by interchanging any two rows or columns of A then

  $det(B) = -det(A)$.

- If B is obtained from A by multiplying one row by a non zero scalar c, then

  $det(B) = c\,det(A)$.

- If B is obtained from A by replacing r_i with $r_i + cr_j$, $i \neq j$, then

  $det(B) = det(A)$.

- If B is singular, in particular if it has a $\vec{0}$ row or column, then $det(B) = 0$

Calculate the determinant of $A = \begin{pmatrix} 0 & 1 & 5 & 0 \\ 3 & -6 & 9 & 0 \\ 2 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$det(A) = -det(\begin{pmatrix} 3 & -6 & 9 & 0 \\ 0 & 1 & 5 & 0 \\ 2 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) =$$

$$= -3det(\begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 2 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) =$$

$$= -3det(\begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 10 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) =$$

$$= -3det(\begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -55 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) =$$

$$= (-3)(-55)det(\begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) =$$

$$= (-3)(-55) = 165$$

det(AB)=det(A)det(B)

If A is a square matrix then det(A^T)=det(A)

If A is invertible then det(A^{-1}) $= \frac{1}{det(A)}$

Check big the orem version 7 in the book