Textbook problems

Section 4.1 :

8) b) Prove $\forall n$ in N 6 div n^3-n

Base case: if $n=1$ then it is true that 6 div 0 since $0 = 0 \cdot 6$

Inductive step: assume 6 div k^3-k, that is $k^3-k=6m$ for some m in N; then $(k+1)^3-(k+1) =$

$= k^3+3k^2+3k+1-k-1 = k^3-k+3(k^2+k) = 6m+3(k^2+k)$ $(*)$

Claim: k^2+k is even; proved at the end.

Using the claim $k^2+k=2t$ for some t in N

and $(k+1)^2-(k+1) = 6m+3\cdot 2t$ (from $(*)$)

$= 6(m+t)$ so 6 div $(k+1)^2-(k+1)$.

Therefore if 6 div k^2-k then 6 div $(k+1)^2-(k+1)$

Proof of claim:

if k is even k^2 is even (proved in class) therefore k^2+k is even

if k is odd $k=2\ell+1$ for some ℓ in N

and $k^2+k = (2\ell+1)^2+(2\ell+1) = 4\ell^2+6\ell+2$

$= 2(2\ell^2+3\ell+1)$ so k^2+k is even

15) c) d) Conjecture $\forall n \geq 1 \ \forall a \in \mathbb{R} \ \frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

Proof

Base case: if $n=1$ and a in $\mathbb{R}$ $\frac{d}{dx} e^{ax} = a e^{ax}$

Inductive step: assume $\frac{d^k}{dx^k}(e^{ak}) = a^k e^{ax}$ for some k in $\mathbb{N}$

then $\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) = \frac{d}{dx}\left(\frac{d^k}{dx^k} e^{ax}\right) = \frac{d}{dx} a^k e^{ax} = a^k \cdot a e^{ax} = a^{k+1} e^{ax}$

18 c) P(1) is true and the base case is proved correctly

The Inductive step correctly proves $P(k) \Rightarrow P(k+1)$ if $k \geq 2$

but the proof does not work for $P(1) \Rightarrow P(2)$

"Assume all dogs in a set of 1 dog have the same breed

and consider $D = \{d_1, d_2\}$ to be a set of two dogs

All dogs in $\{d_2\}$ have the same breed and all dogs in

$\{d_1\}$ have the same breed, but $\{d_1\}$ and $\{d_2\}$ have no

common elements o d_1 and d_2 may have different breed.

Section 4.2

1) c) $\forall n \geq 3 \quad \left(1+\frac{1}{n}\right)^n < n$

Proof

Base case: if $n=3$ $\left(1+\frac{1}{3}\right)^3 = \frac{64}{27} < 3$

Inductive step: assume $\left(1+\frac{1}{k}\right)^k < k$

then $\left(1+\frac{1}{k+1}\right)^{k+1} = \left(1+\frac{1}{k+1}\right)\left(1+\frac{1}{k+1}\right)^k < \left(1+\frac{1}{k+1}\right)\cdot\left(1+\frac{1}{k}\right)^k$

(Since $\frac{1}{k+1} < \frac{1}{k}$)

$< \left(1+\frac{1}{k+1}\right) k$ (by induction assumption)

$= k + \frac{k}{k+1} < k+1$ (since $\frac{k}{k+1} < 1$)

16) $\forall n$ in N $\exists m$ odd $\exists k$ in $N \cup \{0\}$ $n = m\,2^k$

Proof by strong induction:

Base case: If $n=1$ then $m=1$ and $k=0$ and $1 = 1 \cdot 2^0$

Inductive step: assume the statement is true for $n = 1, 2, \dots t$ for some t in N, and consider $t+1$

If $t+1$ is odd then $m = t+1$ and $k=0$ and $t+1 = \underbrace{(t+1)}_{m} 2^{\overset{k}{\overbrace{0}}}$

If $t+1$ is even then $t+1 = 2\ell$ for some ℓ in N and we can use the induction assumption on ℓ:

$\ell = 2^k \cdot m$ with $k \geq 0$ m odd, therefore

$t+1 = 2\ell = 2^{k+1} \cdot m$

Now we need to prove m, k are unique that is

$n = 2^k m = 2^q p \Rightarrow (k = q) \wedge (m = p)$ where k q are non negative integers and m and p are odd

Proof: first we will show $2^k m = 2^q p \Rightarrow k = q$

by contradiction assume $2^k m = 2^q p$ and $k \neq q$.

If $k \neq q$ then one say k is bigger then q

so $2^{k-q} m = p$ but the number on the left is even and the number on the right is odd which is impossible, therefore we must have $k = q$.

Now we will show $(2^k m = 2^q p) \wedge (k = q) \Rightarrow m = p$

$2^k m = 2^m p \Leftrightarrow m = p$ just by algebra

(A proof by strong induction would also work)

Section 4.3

2 b) $f_1 = 1$

$f_2 = 1$

$f_{n+1} = f_n + f_{n-1}$

$\forall n$ in N 5 div f_{5n}

Base case: $f_3 = 2$, $f_4 = 3$, $f_5 = 5$ so 5 div f_5

Inductive step: assume 5 div f_{5k} for some k

$$f_{5(k+1)} = f_{5k+5} = f_{5k+4} + f_{5k+3} = \underbrace{f_{5k+3} + f_{5k+2}}_{f_{5k+4}} + f_{5k+3} =$$

$$= 2(\underbrace{f_{5k+2} + f_{5k+1}}_{f_{5k+3}}) + f_{5k+2} = 3(\underbrace{f_{5k+1} + f_{5k}}_{f_{5k+2}}) + 2f_{5k+1}$$

$$= \underbrace{5f_{5k+1}}_{\text{5 div this}} + \underbrace{3f_{5k}}_{\substack{\text{5 div this} \\ \text{by inductive assumption}}}$$; since by inductive

assumption 5 div f_{5k}, then 5 div $f_{5(k+1)}$

d) $$\sum_{i=1}^{n} f_{2i-1} = f_{2n}$$

Base case: if $n=1$ $\sum_{i=1}^{1} f_{2i-1} = f_1 = 1 = f_{2\cdot 1}$

Inductive step: Assume $\sum_{i=1}^{k} f_{2i-1} = f_{2k}$ for some k

in N, then $\sum_{i=1}^{k+1} f_{2i-1} = \sum_{i=1}^{k} f_{2i-1} + f_{2(k+1)-1} =$

$= f_{2k} + f_{2k+1} = f_{2k+2} = f_{2(k+1)}$

11) $a_1 = 1$

$a_2 = 5$

$a_{n+1} = a_n + 2a_{n-1}$

$\forall n$ in N $a_n = 2^n + (-1)^n$

Proof

Base cases: if $n=1$ $a_1 = 1 = 2^1 + (-1)^1$

if $n=2$ $a_2 = 5 = 2^2 + (-1)^2$

Inductive step: assume $a_k = 2^k + (-1)^k$ and

$a_{k-1} = 2^{k-1} + (-1)^{k-1}$ for some $k \geq 2$

then $a_{k+1} = a_k + 2a_{k-1} = 2^k + (-1)^k + 2\cdot 2^{k-1} + 2\cdot(-1)^{k-1}$

$$= 2^k + 2^k + (-1)^k + 2(-1)^{k-1} = \begin{cases} 2^{k+1} + 1 & \text{if } k \text{ is odd} \\ 2^{k+1} - 1 & \text{if } k \text{ is even} \end{cases}$$

$= 2^{k+1} + (-1)^{k+1}$

Additional problems:

1. Prove that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

   Proof: Let $P(n)$ stand for $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Base case: if $n=1$ $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2}$ and $\frac{1}{1+1} = \frac{1}{2}$, so $P(1)$ is true

Inductive step: Assume $P(k)$ that is $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$, then

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{(k+1)(k+1+1)}$$

therefore $P(k+1)$ is true

2. Let x be a real number, with $x > -1$, and let $P(n)$ stand for $(1+x)^n \geq 1 + nx$. We want to prove $\forall n \geq 0\, P(n)$ by induction on n.

   Base case. . P(0) is $(1+x)^0 \geq 1 + 0x$ which simplifyes to $1 \geq 1$, obviously true. (Note that we start at $n = 0$)

   Induction step:prove $P(k) \Rightarrow P(k+1)$. Assume $P(k)$, prove $P(k+1)$. In detail this means assume: $(1+x)^k \geq 1 + kx$ (and $x > -1$) and prove $(1+x)^{k+1} \geq 1 + (k+1)x$. Start from the induction hypothesis$(1+x)^k \geq (1+kx)$ and multiply both sides of the inequality by $(1+x)$ (remember that $(1+x)$ is positive, this is where we use the hypothesis $x > -1$) to get $(1+x)^k(1+x) \geq (1+kx)(1+x)$ and therefore
   (*) $(1+x)^{k+1} \geq 1+x+kx+kx^2$ (by multiplying out the right hand side). The right hand side of (*) is equal to $1+(k+1)x+kx^2$ and it is greater than or equal to $1+(k+1)x$ since $kx^2 \geq 0$. Therefore $(1+x)^{k+1} \geq 1+(k+1)x$ and we are done.

3. Guess: $3^0 + 3^1 + 3^2 + \cdots + 3^n = \frac{3^{n+1}-1}{2}$. Call this statement $P(n)$.
   Proof by induction:

   - P(0): $3^0 = 1 = \frac{3^1-1}{2}$, therefore P(0) is True.
   - P(k) $\Rightarrow$P(k+1): Assume $3^0 + 3^1 + 3^2 + \cdots + 3^k = \frac{3^{k+1}-1}{2}$ for some $k \geq 0$. (this is P(k)) then $3^0+3^1+3^2+\cdots+3^k+3^{k+1} = \frac{3^{k+1}-1}{2}+3^{k+1}$. (by the induction hypothesis) $= \frac{3 \cdot 3^{k+1}-1}{2} = \frac{3^{k+1+1}-1}{2}$ (reading the first and last term of this chain of equalities we have P(k+1) and we are done).

4. Prove that there are u_{n+1} (the nth Fibonacci number) different ways to tile a 1xn board using squares (i.e. 1x1 tiles) and dominoes (i.e. 1x2 tiles).

   Proof by induction.

   Base cases: if $n = 1$ a 1x1 board can be tiled in only 1 way by using a 1x 1 square; since $u_2 = 1$ the statement is true for $n = 1$.

   if $n = 2$ a 2x1 board can be tiled in 2 ways either by using two 1x 1 squares, or by using a 1x2 tile; since $u_3 = u_1 + u_2 = 2$ the statement is true fr $n = 2$.

   Induction step: assume the statement is true for $n = k$ and $n = k - 1$, consider a 1x (k+1) board: we can start tiling it either by putting down a 1x1 square, in which case we are left with k squares to cover, that is a 1xk board, and by induction assumption we can do it is u_{k+1} ways, or we can start by putting down a domino tile, in which case we are left with k-1 squares to cover, that is a 1x(k-1) board, and by induction assumption we can do it is u_k ways. In total we have $u_{k+1} + u_k = u_{k+2} = u_{k+1+1}$ ways.

5. Consider the following sequence $a_{n\,m}$ where

$$a_{n\,1} = 1 \text{ for all } n \in N$$

$$a_{1\,m} = 0 \text{ for all } m \geq 2$$

$$a_{n+1\,m+1} = a_{n\,m} + a_{n\,m+1} \text{ for all } n,\ m \in N$$

   Prove that $\forall n \in N (x+y)^n = \sum_{i=0}^{n} a_{n+1\,\overset{i+1}{\cancel{i+1}}} x^{n-i} y^i$

a_{1m} .. 1 0 0 0 ...
a_{2m} .. 1 1 0 0 ...
a_{3m} .. 1 2 1 0 ....

In order to prove this we first need to prove

$\forall n \quad a_{nn} = 1 \ \wedge \ \forall m \quad m > n \Rightarrow a_{nm} = 0$

Prove by induction

If $n=1$ $\quad a_{11} = 1$ and $a_{1m} = 0$ for $m > 1$ by definition of a_{nm}

Induction step : assume for some n $\quad a_{nn} = 1$ and $a_{nm} = 0$ if $m > n$

then $\quad a_{n+1\,n+1} = a_{n\,n} + a_{n\,n+1} = 1 + 0 = 1$ and if $m > n+1$ then $m-1 > n$ so

$a_{n+1\,m} = a_{n\,m-1} + a_{n\,m} = 0$

a_{1m} ·· 1 0 0 0 ···
a_{2m} ·· 1 1 0 0 ···
a_{3m} ·· 1 2 1 0 ····

Proof of $(x+y)^n = \sum_{l=0}^{n} a_{n+1\ l+1}\, x^{n-l} y^l$ by induction

Base case: if $n=1$ $\quad (x+y) = \sum_{l=0}^{1} a_{2\ l+1}\, x^{1-l} y^l = a_{21}\, x + a_{22}\, y = x+y$

Induction step: assume $(x+y)^k = \sum_{l=0}^{k} a_{n+1\ l+1}\, x^{n-l} y^l$

then $(x+y)^{k+1} = (x+y)(x+y)^k = (x+y) \sum_{l=0}^{k} a_{k+1\ l+1}\, x^{k-l} y^l =$

$\sum_{l=0}^{k} a_{k+1\ l+1}\, x^{k+1-l} y^l + \sum_{l=0}^{k} a_{k+1\ l+1}\, x^{k-l} y^{l+1}$; $\quad x^{k-l} y^{l+1} = x^{k+1-(l+1)} y^{l+1}$

$= \sum_{l=0}^{k} a_{k+1\ l+1}\, x^{k+1-l} y^l + \sum_{l=1}^{k+1} a_{k+1\ l}\, x^{k+1-l} y^l$

$= \underbrace{a_{k+1\ 1}}_{= a_{k+2\ 1}} x^{k+1} + \sum_{l=1}^{k} (\underbrace{a_{k+1\ l+1} + a_{k+1\ l}}_{= a_{k+2\ l+1}})\, x^{k+1-l} y^l + \underbrace{a_{k+1\ k+1}}_{= 1} y^{k+1}$ $\quad = a_{k+2}\, a_{k+2}$

$\sum_{l=0}^{k+1} a_{k+2\ l+1}\, x^{k+1-l} y^l$

(If the algebra above is confusing you can try to write down all the sums so $\sum_{l=0}^{k} a_{k+1\ l+1}\, x^{k+1-l} y^l + \sum_{l=0}^{k} a_{k+1\ l+1}\, x^{k-l} y^{l+1} =$

$a_{k+1,1}\, x^{k+1} + a_{k+1\ 2}\, x^k y + a_{k+1\ 3}\, x^{k-1} y^2 + \dots + a_{k+1\ k+1}\, x\, y^k +$

$+ a_{k+1\ 1}\, x^k y + a_{k+1\ 2}\, x^{k-1} y^2 + \dots + a_{k+1\ k}\, x\, y^k + a_{k+1\ k+1}\, y^{k+1}$