- **Problem 1** In this problem 3 div a means *3 divides a* and 3 notdiv a means *3 does not divide a.*

  - Write a statement equivalent to the negation of

    $$\forall x \in Z,\, \forall y \in Z,\ 3 \text{ div } xy \Leftrightarrow (3 \text{ div } x \vee 3 \text{ div } y)$$

    that does not contain the negation symbol ¬ (not). You are allowed to use notdiv.

$\exists x \in Z\ \exists y \in Z\ (3 \text{div}\, xy \wedge 3 \text{ notdiv } x \wedge 3 \text{ notdiv } y) \vee (3 \text{ notdiv } xy \wedge (3 \text{ div } x \vee 3 \text{ div } y)$

  - Prove or disprove that

    $$\forall x \in Z,\, \forall y \in Z,\ 3 \text{ div } xy \Leftrightarrow (3 \text{ div } x \vee 3 \text{ div } y)$$

True

Proof of =>: by contraposition assume 3 notdiv x $\wedge$ 3 notdiv y.

Then $x = 3q_1 + r_1$ $r_1 = 1$ or 2 and $y = 3q_2 + r_2$ $r_2 = 1$ or 2

so $xy = (3q_1 + r_1)(3q_2 + r_2) = 9q_1q_2 + 3q_1r_2 + 3q_2r_1 + r_1r_2 = 3(3q_1q_2 + q_1r_2 + q_2r_1) + r_1r_2$

r_1r_2 can be 1, 2, or 4 if $r_1r_2 = 4$ we can write $xy = 3(3q_1q_2 + q_1r_2 + q_2r_1 + 1) + 1$

in any case when r_1r_2 is divided by 3 the remainder is not 0

so 3 notdiv xy

Proof of <= assume 3 div x $\vee$ 3 div y; if 3 div x then $x = 3k$ for some k in Z and $xy = 3ky$ so 3 div xy; if 3 div y the argument is similar

  - Prove or disprove that

    $$\forall x \in Z,\, \forall y \in Z,\ 3 \text{ div } xy \Leftrightarrow (3 \text{ div } x \wedge 3 \text{ div } y)$$

False: if $x = 2$ and $y = 3$ 3 div $x \cdot y$ is true but 3 div $x \wedge$ 3 div y is false

Scratch work

x	$f(x)$	x	$f(x)$	x	$f(x)$
-1	1	1	0	2	-1
-2	0	3	2	4	1
-3	-1	5	4	6	3

- **Problem 2** Define a function $f : Z \to Z$ by:

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ x-1 & \text{if } x \text{ is odd and } x \geq 0 \\ x-3 & \text{if } x \text{ is even and } x \geq 0 \end{cases}$$

a) Is f injective ? *(Give a proof).*

No $f(-2) = f(1)$

b) Is f surjective ? *(Give a proof).*

yes. We need to prove $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ y = f(x)$

given y in $\mathbb{Z}$ if $y \leq 1$ take $x = y-2$ then $x < 0$ and $f(x) = x+2 = y$

if $y > 1$ and even take $x = y+1$, then $x > 0$ and x odd so

$f(x) = x-1 = y$

if $y > 1$ and y odd take $x = y+3$, then $x > 0$ and x is even so $f(x) = x+3 = y$

- **Problem 3** Prove that $\sum_{i=1}^{2n}(-1)^i i = n$

Proof by induction on N

Base case: If $n=1$ $\sum_{i=1}^{2}(-1)^i \cdot i = -1+2 = 1$

Induction step: assume $\sum_{i=1}^{2k}(-1)^i \cdot i = k$ for some $k \geq 1$; then

$$\sum_{i=1}^{2(k+1)}(-1)^i \cdot i = \sum_{i=1}^{2k}(-1)^i \cdot i + (-1)^{2k+1}(2k+1) + (-1)^{2k+2}(2k+2) = k-(2k+1)+(2k+2) = k+1$$

- **Problem 4** For each of the following statements circle whether the statement is true or false and give a proof.

1. $\exists x \in Z, \quad \forall y \in Z, \ x - y = 10$.

   TRUE FALSE

Prove $\forall x \in Z \ \exists y \in Z \ x-y \neq 10$

Given x in Z take $y=x$, then $x-y=0 \neq 10$

2. $\forall x \in Z, \quad \exists S \in P(Z), \forall w \in S, \ x + w = 0$ (Here $P(Z)$ is the power set of Z) .

   TRUE FALSE

Given x in Z take $S=\{-x\}$ then if $w \in S$ $w=-x$

and $x+w = x+(-x) = 0$

- **Problem 5** Let A, B , C be sets. Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

First prove $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$
assume $x \in A \times (B \cap C)$ then $x = (a, d)$ $a \in A$, $d \in B \cap C$ so $d \in B$ and
$d \in C$ so $x \in A \times B$ and $x \in A \times C$ so $x \in A \times B \cap A \times C$
Then prove $(A \times B) \cap (A \times C) \leq A \times (B \cap C)$: assume $x \in (A \times B) \cap (A \times C)$ then $x \in (A \times B)$
and $x \in A \times C$ so $x = (a, d)$ $a \in A$ $d \in B$ and $d \in C$ so $d \in B \cap C$ so
$x \in A \times (B \cap C)$