Spring 2018 Math 300 Final exam

Write clearly and legibly. Justify all your answers.

You will be graded for correctness and clarity of your solutions.

You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.

You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.

Please raise your hand and ask a question if anything is not clear.

This exam contains 9 pages and is worth a total of 70 points. You have 1 hr and 50 minutes. Good luck

NAME:_________

PROBLEM 1 (8 points) _______

PROBLEM 2 (8 points) _______

PROBLEM 3 (12 points) ________

PROBLEM 4 (8 points) ________

PROBLEM 5 (8 points) ________

PROBLEM 6 (8 points) ________

PROBLEM 7 (10 points) ________

PROBLEM 8 (8 points) ________

Total ____________

- **Problem 1** Given sets A, B, C in some universe U prove that
$$((A \cup B) - C)^c = (A - C)^c \cap (B - C)^c.$$

First we shall prove $((A \cup B) - C)^c \subseteq (A - C)^c \cap (B - C)^c$

assume $x \in ((A \cup B) - C)^c$, then $x \notin (A \cup B) - C$ so $x \notin A \cup B \vee x \in C$

if $x \notin A \cup B$ then $x \notin A \wedge x \notin B$ so $x \notin A - C$ and $x \notin B - C$

so $x \in (A - C)^c$ and $x \in (B - C)^c$ and therefore $x \in (A - B)^c \cap (B - C)^c$.

if $x \in C$ again $x \notin A - C$ and $x \notin B - C$ so $x \in (A - C)^c$ and $x \in (B - C)^c$

and therefore $x \in (A - C)^c \wedge (B - C)^c$

Now we shall prove $(A - C)^c \cap (B - C)^c \subseteq ((A \cup B) - C)^c$

Assume $x \in (A - C)^c \cap (B - C)^c$ then $x \notin A - C$ and $x \notin B - C$

If $x \in C$ then $x \notin (A \cup B) - C$ so $x \in ((A \cup B) - C)^c$

If $x \notin C$ then it must be the case that $x \notin A$ and $x \notin B$

so $x \notin A \cup B$ so $x \notin A \cup B - C$ and $x \in (A \cup B - C)^c$

- **Problem 2** Write a statement equivalent to the negation of

$$\forall x \in A\, \exists y \in B\, (x > y) \Rightarrow (\exists z \in C\big((z > x) \Rightarrow (z > y \vee z = 0)\big)\big)$$

that does not use the negation symbol $\neg$

My mistake for not using enough parentheses

Some people read it as

$\forall x \in A\ \exists y \in B\ (x > y \Rightarrow \cdots\cdots\cdots)$

Negation

$\exists x \in A\ \forall y \in B\big(x > y \wedge \forall z \in C\ (z > x \wedge z \leq y \wedge z \neq 0)\big)$

Some people read it as

$(\forall x \in A\ \exists y \in B\ x > y) \Rightarrow (\exists z \in C\ (z > x \Rightarrow (z > y \vee z = 0))$

Negation

$\forall x \in A\ \exists y \in B\ x > y \wedge \forall z \in C\ (z > x \wedge z \leq y \wedge z \neq 0)$

Both received full credit

- **Problem 3** Prove that $\overbrace{3 \text{ div } x}^{\forall x \in \mathbb{Z}} \Leftrightarrow 3 \text{ div } x^3$

First prove $3 \text{ div } x \Rightarrow 3 \text{ div } x^3$

Assume $x \in \mathbb{Z}$ and $3 \text{ div } x$, then $x = 3k$ for some $k \in \mathbb{Z}$ and $x^3 = 27k^3 = 3(9k^3)$ so $3 \text{ div } x^3$

Then prove $3 \text{ div } x^3 \Rightarrow 3 \text{ div } x$

By contraposition assume $x \in \mathbb{Z}$ and 3 does not divide x, then $x \equiv 1$ or $x \equiv 2 \mod 3$ and therefore $x^3 \equiv 1$ or $2 \mod 3$ so 3 does not divide x^3

Alternative proof: 3 is prime assume $3 \text{ div } x \cdot x \cdot x$ then 3 has to divide one of the factors in the product $x \cdot x \cdot x$ so $3 \text{ div } x$

Use the result above to prove that $\sqrt[3]{3}$ is irrational

By contradiction assume $\sqrt[3]{3} = \frac{m}{n}$ with $\frac{m}{n}$ reduced then $3n^3 = m^3$ so $3 \text{ div } m^3$ and therefore $3 \text{ div } m$, so $m = 3k$ for some $k \in \mathbb{Z}$ and $3n^3 = (3k)^3 = 27k^3$ so $n^3 = 3(3k^3)$ and $3 \text{ div } n^3$ and therefore n. so 3 is a common factor of m and n and therefore $\frac{m}{n}$ is not reduced, contradiction.

- **Problem 4** Prove that the set A $=\{S | S \subseteq N$ and $|S| \leq 1\}$ (that is A is the set of all subsets of N that have zero or one element) is denumerable.

$A = \{ \emptyset, \{1\}, \{2\}, \{3\}, \ldots\ldots \}$

Define $f \quad N \to A$

$$f(n) = \begin{cases} \emptyset & \text{if } n=1 \\ \{n-1\} & \text{otherwise} \end{cases}$$

Define $g \quad A \to N$

$$g(S) = \begin{cases} 1 & \text{if } S = \emptyset \\ n+1 & \text{if } S = \{n\} \end{cases}$$

$$g(f(n)) = \begin{cases} g(\emptyset) = 1 & \text{if } n=1 \\ g(\{n-1\}) = n & \text{if } n>1 \end{cases}$$ so $\forall n \in N \quad g(f(n)) = n$

$$f(g(S)) = \begin{cases} f(1) = \emptyset & \text{if } S = \emptyset \\ f(n+1) = \{n\} = S & \text{if } S = \{n\} \end{cases}$$ so $\forall S \in A \quad f(g(S)) = S$

Therefore $g = f^{-1}$ and f is a bijection, so A is denumerable

- **Problem 5** Prove that if p and q are distinct primes then
  $\forall a \in Z, \forall b \in z,\ a \equiv b \bmod pq \Leftrightarrow (a \equiv b \bmod p \wedge a \equiv b \text{mod } q)$

Given $a, b \in \mathbb{Z}$

First prove $a \equiv b \bmod pq \Rightarrow (a \equiv b \bmod p \wedge a \equiv b \bmod q)$

Assume $a \equiv b \bmod pq$ then pq div $a-b$ so
$a-b = pq \cdot k$ for some $k \in \mathbb{Z}$, so p div $a-b$ and therefore $a \equiv b \bmod p$
and q div $a-b$ so $a \equiv b \bmod q$

Now prove $a \equiv b \bmod p \wedge a \equiv b \bmod q \Rightarrow a \equiv b \bmod pq$
Assume p div$(a-b)$ and q div $(a-b)$ so $a-b = kp = hq$ for some $h, k \in \mathbb{Z}$
Then $kp = hq$ so p div hq and p does not divide q, so
p divides h, therefore $h = s\underline{p}$ for some $s \in \mathbb{Z}$
and $a - b = spq$ so pq div $a-b$ and therefore $a \equiv b \bmod pq$

Give a counterexample to show this theorem is not true if p and q are not prime

$6 \equiv 2 \bmod 4$ $\quad$ $6 \equiv 2 \bmod 2$ $\quad$ but $6 \not\equiv 2 \bmod 8$

- **Problem 6** Find all integer solutions of $3^{122}x \equiv 5 \bmod 11$

11 is prime $3^{10} \equiv 1 \bmod 11$ $3^{122} = 3^2 (3^{10})^{12} \equiv 9 \bmod 11$

$9x \equiv 5 \bmod 11$ has only on solution in $\mathbb{Z}_{11}$ by trial and error $x = 3$

So $\boxed{x = 3 + k11}$

- **Problem 7** Consider the relation R on Z defined by

$$aRb \text{ iff } 3a + 3b \equiv 0 \bmod 6$$

Prove R is an equivalence relation.

aRa since $3a+3a = 6a \equiv 0 \bmod 6$

$aRb \Rightarrow bRa$ since if $3a+3b \equiv 0 \bmod 6$ then $3b+3a \equiv 0 \bmod 6$

$aRb \wedge bRc \Rightarrow aRc$ since $3a+3b \equiv 0 \bmod 6$ and $3b+3c \equiv 0 \bmod 6$

implies $(3a+3b)+(3b+3c) = 3a+6b+3c \equiv 3a+3c \equiv 0 \bmod 6$

List all equivalence classes for R

$b \in [a]$ iff $3a+3b \equiv 0 \bmod 6$ iff $3a \equiv -3b \bmod 6$ iff

$3a \equiv 3b \bmod 6$ iff $a \equiv b \bmod 2$

So there are only 2 equivalence classes

$[0] =$ EVEN

$[1] =$ ODD

- **Problem 8** Consider the sequence $\{a_n\}$ defined by:

  $a_1 = 2$

  $a_2 = 3$

  $a_3 = 2$

  $a_{n+1} = 4a_n - 5a_{n-1} + 2a_{n-2}$ if $n + 1 \geq 4$

  Prove that $\forall n \geq 1$, $a_n = 3n + 1 - 2^n$

By induction

Base case: if $n = 1$ $\quad 3+1-2 = 2 = a_1$

if $n = 2$ $\quad 6+1-4 = 3 = a_2$

if $n = 3$ $\quad 9+1-8 = 2 = a_3$

Induction step: assume the formula true for a_{k-2}, a_{k-1}, a_k for some $k \geq 3$

then
$$a_{k+1} = 4(3k+1-2^k) - 5(3(k-1)+1-2^{k-1}) + 2(3(k-2)+1-2^{k-2}) =$$
$$= 12k + 4 - 2 \cdot 2^{k+1} - 15k + 15 - 5 + 5 \cdot 2^{k-1} + 6k - 12 + 2 - 2^{k-1} =$$
$$= 3k + 4 - 2 \cdot 2^{k+1} + 4 \cdot 2^{k-1} =$$
$$= 3(k+1) + 1 - 2^{k+1}$$