Math 300 Spring 2017 Midterm Exam

Write clearly and legibly. Justify all your answers.
You will be graded for correctness and clarity of your solutions.
You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.
You can use elementary algebra and any result that we proved in class. You need to prove everything else.
Please raise your hand and ask a question if anything is not clear.
This exam contains 5 pages and is worth a total of 50 points. You have 50 minutes. Good luck

NAME:_________

PROBLEM 1 (10) _______

PROBLEM 2(10) _______

PROBLEM 3 (10) ________

PROBLEM 4 (10)________

PROBLEM 5 (10) ________

Total ____________

Problem 1: Let A and B, C be sets.

1. (5 points) Prove that $(A - B) \cap (A - C) \subseteq A - (B \cap C)$

Assume $x \in (A-B) \cap (A-C)$ then $x \in (A-B)$ and $x \in (A-C)$ Therefore $x \in A$ and $x \notin B$ and $x \notin C$ so $x \in A$ and $x \notin B \cap C$ so $x \in A - (B \cap C)$

2. (7 points) Is $\forall A, B, C \quad (A - B) \cap (A - C) = A - (B \cap C)$ true ? Justify your answer.

No take $A = \{1, 2\}$ $B = \{1\}$ $C = \{2\}$

$(A - B) \cap (A - C) = \{2\} \cap \{1\} = \emptyset$

$A - \{B \cap C\} = \{1, 2\}$

It is true that $(A - B) \cap (A - C) = A - (B \cup C)$

Proof $x \in (A-B) \cap (A-C) \Leftrightarrow x \in A \wedge x \notin B \wedge x \notin C \Leftrightarrow x \in A \wedge x \notin (B \cup C) \Leftrightarrow x \in A - (B \cup C)$

Problem 2 (10 points) Prove that $\forall x \in Z$ 14 div $x \Leftrightarrow$ (2 div $x \wedge 7$ div x)

$\Rightarrow$ Assume 14 div x, then $x = 14k = 2 \cdot 7k$ for some $k \in Z$ so 2 div x and 7 div x

$\Leftarrow$ Assume 2 div x and 7 div x, then $x = 2h = 7k$ for some $h, k \in Z$. Since $2h$ is even $7k$ must be even, so k must be even, that is $k = 2\ell$ for some $\ell \in Z$ so $x = 7 \cdot 2\ell = 14\ell$ and 14 div x

Problem 3(10 points) Guess a formula for $1 + 3 + 5 + \cdots (2n+1)$, the sum of the first n odd positive integers and use induction to prove your formula is correct.

1
$1+3=4$
$1+3+5=9$
$1+3+5+7=16$
$1+3+5+7+9=25$

Guess $\sum_{i=1}^{n} (2i-1) = n^2$

Proof by induction

Base case: if $n=1$ $\sum_{i=1}^{1} (2i-1) = 1 = 1^2$

Induction step: assume $\sum_{i=1}^{k} (2i+1) = k^2$ then

$$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=0}^{k} (2i-1) + 2(k+1) - 1 = k^2 + 2k + 1 = (k+1)^2$$

Problem 4 Define a function $f : Z \to Z$ by:

$$f(x) = \begin{cases} x - 3 & \text{if } x \geq 0 \\ x + 5 & \text{if } x < 0 \end{cases}$$

1. (5 points) Is f injective ? Prove your answer.

No $f(0) = -3$
$f(-8) = -3$

2. (5 points) Is f surjective ? Prove your answer.

Yes We need to proce $\forall y \in Z \; \exists x \in Z \; f(x) = y$

if $y \geq -3$ take $x = y+3$ then $x \geq 0$ and $f(x) = x-3 = y+3-3 = y$

if $y < -3$ take $x = y-5$ then $x < 0$ and $f(x) = x+5 = y-5+5 = y$

Problem 5(10 points) Let A be the set of all functions from Z to Z. For each statement below , write the negation of the statement and prove whether the original statement(NOT the negation) is true or false.

(a) $\forall f \in A\, \exists g \in A\, \forall x \in Z\; g(x) \geq f(x)$.
NEGATION: $\exists f \in A \;\; \forall g \in A \;\; \exists x \in Z \;\; g(x) < f(x)$

True or false ? Give a proof.

True Given f $Z \to Z$ teke g $Z \to Z$
$g(x) = f(x)$ then $\forall x \in Z \;\; g(x) \geq f(x)$

(b) $\exists f \in A\, \forall g \in A\, \forall x \in Z\; g(x) \geq f(x)$.
NEGATION: $\forall f \in A \;\; \exists g \in A \;\; \exists x \in Z \;\; g(x) < f(x)$

True or false ? Give a proof.

Felse. The negetion is true.
Given f $Z \to Z$ Teke g $Z \to Z$ defined by
$g(x) = f(x) - 1$ teke $x = 0$ then $g(0) = f(0) - 1$ so
$g(0) < f(0)$

(c) $\exists f \in A\, \forall y \in Z\, \exists x \big(y \text{ odd} \Rightarrow (x \text{ even} \wedge f(x) = y).\big)$
NEGATION:
$\forall f \in A \;\; \exists y \in Z \;\; \forall x \quad y \text{ odd} \wedge \; (x \text{ odd} \vee f(x) \neq y)$
True or false ? Give a proof.

True Teke f $Z \to Z$
$f(x) = x + 1$
Tlen if y is odd $x = y - 1$ is even end $f(x) = y$