Math 300 Spring 2017 Final Exam

You can use any result we proved in class and elementary algebra. You need to prove everything else.
Write clearly and legibly. Prove all your answers.
You will be graded for correctness and clarity of your solutions.
You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.
This exam contains 9 pages and is worth a total of 80 points.
You have one hour and 50 minutes. Good luck

FIRST AND LAST NAME:__________

PROBLEM 1 _______

PROBLEM 2 _______

PROBLEM 3 ________

PROBLEM 4 ________

PROBLEM 5 ________

PROBLEM 6 ________

PROBLEM 7 ________

PROBLEM 8 ________

Total ____________

Problem 1 (10 points)Prove that , $\forall x \in Z$, 7 div $x \Leftrightarrow$ 7 div x^2.

we shall prove $x \equiv 0 \mod 7 \iff x^2 \equiv 0 \mod 7$

$\Rightarrow$ Assume $x \equiv 0 \mod 7$ then $x^2 \equiv 0 \cdot 0 = 0 \mod 7$

$\Leftarrow$ By contraposition assume $x \not\equiv 0 \mod 7$ then either

$x \equiv 1$ so $x^2 \equiv 1 \mod 7$ or
$x \equiv 2$ so $x^2 \equiv 4 \mod 7$ or
$x \equiv 3$ so $x^2 \equiv 2 \mod 7$ or
$x \equiv 4$ so $x^2 \equiv 2 \mod 7$ or
$x \equiv 5$ so $x^2 \equiv 4 \mod 7$ or
$x \equiv 6$ so $x^2 \equiv 1 \mod 7$

in any case $x^2 \not\equiv 0 \mod 7$

Problem 2(10 points) Compute $(20 * (16)^{40} + 13^{1600}) \bmod 17$.

$3 \cdot (-1)^{40} + ((13)^{16})^{100}$ 17 is prime so $13^{16} \equiv 1 \mod 17$

$3 + 1^{100} = \boxed{4}$

Find the unit digit of 7^{1000}.

We need to compute $7^{1000} \mod 10$

$7^{1000} = (49)^{500} \equiv (-1)^{500} \equiv \boxed{1} \mod 7$

Problem 3 (10 points) Let EVEN= $\{x \in Z^+ \mid x \text{ is even }\}$ and ODD=$\{x \in Z^+ \mid x \text{ is odd }\}$. Prove EVEN $\times$ ODD is denumerable. You may assume Z, EVEN, ODD, $Z \times Z$, $Z^+ \times Z^+$ are denumerable. ~~You may also assume that the composition of two bijections is a bijection.~~

Consider $f \quad Z^+ \times Z^+ \to EVEN \times ODD$

$f((x,y)) = (2x, 2y-1)$

$g \quad EVEN \times ODD \to Z^+ \times Z^+$

$g(a \quad b) = \left(\frac{a}{2} \quad \frac{b+1}{2}\right)$

then $g(f((x\ y)) = g(2x, 2y-1) = \left(\frac{2x}{2}, \frac{2y-1+1}{2}\right) = (x,y)$

$f(g((a,b))) = f\left(\frac{a}{2}, \frac{b+1}{2}\right) = \left(2\frac{a}{2}, 2\frac{(b+1)}{2} - 1\right) = (a,b)$

therefore $g = f^{-1}$, so f is a bijection

and EVEN $\times$ ODD is equipotent to a denumerable set, so it is denumerable.

Problem 4 (10 points) Prove that a every prime number greater than 3 is of the form 6n+1 or 6n -1 for some n in Z^+.

Suppose $p > 3$ p is prime; by the division theorem

$p = 6q + r$ for some $q \in Z$, $r \in Z$, $0 \le r \le 5$

If $r = 0, 2, 4$ then 2 div p so $\underline{p}$ is not prime

If $r = 3$ then 3 div p so $\underline{p}$ is not prime

therefore the only possible remainders are 1 or 5

so $p = 6q+1$ or $p = 6q+5 = 6(q+1) - 1$

Since $p > 0$ $\underline{q} \ge 0$ if $\underline{q} = 0$ $q+1 > 0$

so if $p > 0$ and p is prime $p = 6q+1$ and $q \in Z^+$ or

$p = 6(q+1) - 1$ and $q+1 \in Z^+$

and we have primes of both forms, for example

$5 = 6 \cdot 1 - 1$, $7 = 6 \cdot 1 + 1$

Problem 5 (10 points) List all the elements of $P(\{1\} \times \{a, b\})$.

$\{1\} \times \{a, b\} = ((1, a), (1, b))$

$P(\{1\} \times \{a, b\}) = \{\emptyset, \{(1, a)\}, \{(1, b)\}, \{(1, a), (1, b)\}\}$

Problem 6 (10 points) Find all integers x that satisfy $36x \equiv 30 \bmod 42$

$6x \equiv 5 \mod 7$

$6x \equiv 5 \mod 7$

By trial end error $x = 2$

so $x = 2 + 7k \quad k \in \mathbb{Z}$

Find the inverse of 5 in Z_7.

By trial end error $5^{-1} = 3$ in $\mathbb{Z}_7$

since $5 \cdot 3 \equiv 1 \mod 7$

Problem 7(10 points) Remember from calculus that the $(\sin(x))' = \cos(x)$ and $(\cos(x))' = -\sin(x)$, that is the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$. Prove that for any $n \in \mathbb{Z}^+$ the nth derivative of $\sin(x)$ is given by the formula :

$$(\sin(x))^{(n)} = \begin{cases} \overset{\cos x}{\cancel{\sin(x)}} & \text{if } n \equiv 1 \bmod 4 \\ -\sin(x) & \text{if } n \equiv 2 \bmod 4 \\ -\cos(x) & \text{if } n \equiv 3 \bmod 4 \\ +\overset{\sin x}{\cancel{\cos(x)}} & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

By induction:

if $n = 1$ $\quad \sin(x) = \cos x$

Induction step:

Assume the formula is true for some $k \geq 1$. We need to show it is true for $k+1$.

if $k+1 \equiv 2 \bmod 4$ $\quad k \equiv 1 \bmod 4$ and $(\sin(x))^{k+1} = (\sin(x)^k)'$
$= (\cos x)' = -\sin x$

if $k+1 \equiv 3$ then $k \equiv 2 \bmod 4$ and $(\sin(x))^{k+1} = (-\sin x)' = -\cos x$

if $k+1 \equiv 0 \bmod 4$ then $k \equiv 3 \bmod 4$ and $(\sin(x))^{k+1} = (-\cos x)' =$
$= \sin x$

if $k+1 \equiv 1 \bmod 4$ then $k \equiv 0 \bmod 4$ and $(\sin(x))^{k+1} = (\sin x)' =$
$\cos x$

There fore the formule holds for $k+1$

Problem 8 Let R be the relation on Z defined by aRb iff $a - b$ is even. Prove R is an equivalence relation and describe its equivalence classes.

$\forall a \in \mathbb{Z}$ $a-a=0$ which is even, so aRb
$\forall a\ b \in \mathbb{Z}$ if $a-b$ is even then $a-b = 2k$ for some $k \in \mathbb{Z}$
so $b-a = -(a-b) = 2(-k)$ so $b-a$ is even
therefore $aRb \Rightarrow bRa$
$\forall a\ b\ c \in \mathbb{Z}$ $a-b$ is even and $b-c$ is even implies
$a-c = (a-b)+(b-c)$ is even so aRb and $bRc \Rightarrow$
aRc

There are two equivalence classes
$[1]_R = ODD_{\mathbb{Z}}$
$[0]_R = EVEN_{\mathbb{Z}}$