Problem 1 (10 points) Assume P, Q and R are statements. Is $\neg P \vee \neg Q \vee R$ equivalent to $P \Rightarrow (Q \Rightarrow R)$? Justify your answer.

yes : $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $\neg P \vee (Q \Rightarrow R)$ which is equivalent to $\neg P \vee \neg Q \vee R$

Alternatively we can use truth tables:

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$\neg P \vee \neg Q \vee R$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
F	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Problem 2(10 points) Given the following recursive definition:

$a_1 = 1$

$a_{n+1} = 3a_n - 1$ for $n \geq 1$

Prove that $a_n = \frac{3^{n-1}+1}{2}$

By induction on n

Base case: if $n=1$ $a_1 = 1 = \dfrac{3^{1-1}+1}{2}$

Induction step: assume $a_n = \dfrac{3^{n-1}+1}{2}$ then

$$a_{n+1} = 3a_n - 1 = 3 \cdot \frac{3^{n-1}+1}{2} - 1 = \frac{3^n + \underline{3 - 2}}{2} = \frac{3^{(n+1)-1}+1}{2}$$

Problem 3 (10 points) Assume A is a denumerable set .
Let $B = \{x \in Z^+ \mid x \geq 10\}$ Prove $A \cup B$ is denumerable. You may assume A and B are disjoint. You may also assume Z, $Even_{Z^+}$, $Even_Z$, Odd_{Z^+}, Odd_Z, $Z \times Z$, $Z^+ \times Z^+$ are denumerable, but you cannot assume the result from the hw that the union of two denumerable sets is denumerable without reproving it.

Let $f\ Z^+ \to A$ be a bijection. Such a bijection exists because A is denumerable.

Let $g\ Z \to A \cup B$

$$g(z) = \begin{cases} f(z) & \text{if } z \geq 1 \\ -z+10 & \text{if } z \leq 0 \end{cases}$$

Let $h\ A \cup B \to Z$

$$h(y) = \begin{cases} f^{-1}(y) & \text{if } y \in A \\ 10-y & \text{if } y \in B \end{cases}$$

since $A \cap B = \phi$ h is well defined

$$h(g(z)) = \begin{cases} h(\underset{\in A}{f(z)}) = f^{-1}f(z) = z & \text{if } z \geq 1 \\ h(\underset{\in B}{-z+10}) = 10-(-z+10) = z & \text{if } z \leq 0 \end{cases}$$

$$g(h(y)) = \begin{cases} g(\underset{\in Z^+}{f^{-1}(y)}) = f(f^{-1}(y)) = y & \text{if } y \in A \\ g(\underset{\leq 0}{10-y}) = -10+y+10 = y & \text{if } y \in B \end{cases}$$

In any case $h(g(z)) = z$ for all $z \in Z$ and
$g(h(y)) = y$ for all $y \in A \cup B$

so $h = g^{-1}$ and g is a bijection from a denumerable set to $A \cup B$, so $A \cup B$ is denumerable

Problem 4 (10 points) Prove that no integer of the form 4n+3 is the sum of two squares (that is the sum of the squares of two integers).

We need to prove the diophentine equation $4n+3 = x^2+y^2$ has no solutions.

By contradiction assume $4n_0+3 \equiv x_0^2+y_0^2$ for some $n_0, x_0, y_0 \in \mathbb{Z}$ then $4n_0+3 \equiv x_0^2+y_0^2 \mod 4$ or $3 \equiv x_0^2+y_0^2 \mod 4$

but in $\mathbb{Z}_4$
$0^2=0$
$1^2=1$
$2^2=0$
$3^2=1$

so $x_0^2+y^2 \mod 4$ can only be 0, 1 or 2, never 3

Problem 5 (10 points) Prove that for any sets A,B and C
$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

First we shall prove $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$:
assume $x \in A \times (B \cap C)$ then $x = (a, d)$, with $a \in A$ $d \in B \cap C$
so $d \in B$ and $x \in A \times B$ and $d \in C$ so $x \in A \times C$, therefore
$x \in (A \times B) \cap (A \times C)$

Then we shall prove $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$:
assume $y \in (A \times B) \cap (A \times C)$: then $y \in A \times B$ so $y = (a, d)$ with
$a \in A$, $d \in B$; $y \in A \times C$ also so we must have $d \in C$, therefore
$d \in B \cap C$ so $y \in A \times (B \cap C)$

Problem 6
(5 points) Find all integers x that satisfy $12x \equiv 24 \bmod 30$.

$2x \equiv 4 \mod 5$

$x \equiv 2 \mod 5$

$x = 2 + 5k \quad k \in \mathbb{Z}$

(5 points) Compute $2^{55} \bmod 53$. (Note : 53 is prime)

$2^{55} = 2^{53} \cdot 2^2 \equiv 2 \cdot 4 \equiv 8 \mod 53$

(5 points) List all invertible elements of Z_{18}.

a is invertible in Z_{18} if $(a, 18) = 1$ so

1, 5, 7, 11, 13, 17

Problem 7(10 points) Given $f: A \to A$ and $g: A \to A$.

1. Prove that if f and g are both injective, then $g \circ f$ is injective

Assume $g(f(x_1)) = g(f(x_2))$ then $f(x_1) = f(x_2)$ because g is injective and therefore $x_1 = x_2$ because f is injective

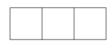
2. (5 points) Is it true that if $g \circ f$ is injective then f and g are both injective? Prove your answer.

No Consider $f \; \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x) = x+1$ $\quad g \; \mathbb{Z}^+ \to \mathbb{Z}^+$, $g(x) = \begin{cases} 2 & \text{if } x = 1 \\ x & \text{if } x > 1 \end{cases}$

g is not injective but $g \circ f \; \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by $g(f(x)) = g(x+1) = x+1$ so it is injective

Problem 8 Prove that an 8x8 checkerboard with the square in position 1, 2 i.e top row , second from the left removed cannot be covered by 1x3 tiles. A covering must be such that each square of a 1x3 tiles covers exactly one square of the board.

B	■	Y	B	R	Y	B	R
Y	B	R	Y	B	R	Y	B
R	Y	B	R	Y	B	R	Y
B	R	Y	B	R	Y	B	R
Y	B	R	Y	B	R	Y	B
R	Y	B	R	Y	B	R	Y
B	R	Y	B	R	Y	B	R
Y	B	R	Y	B	R	Y	B



1x3 tile

checkerboard with square (1,2) removed

Assume by contradiction that it is possible to cover the board.
Color the board with 3 colors B, R, Y as above. A 1x3 tile placed anywhere on the board will cover exactly one B, one R one Y square. So the total number of B R Y squares must be the same but there are 22 B 20 R 21 Y squares on the board