Problem 1 (12 points) Prove that if $a, b$ and $c$ are integers and $a^2 + b^2 = c^2$ then $a$ is even or $b$ is even.

By contradiction; assume $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$ and $a$ is odd and $b$ is odd. Then $a = 2k + 1$ for some $k \in \mathbb{Z}$ and $b = 2h + 1$ for some $h \in \mathbb{Z}$.

$$e^2 + b^2 = (2k + 1)^2 + (2h + 1)^2 = 4(k^2 + h^2 + h + k) + 2$$

so $2 \div e^2 + b^2$ but $4$ does not divide $c^2$. Since $2 \div c^2$ then $2 \div c$ so $c = 2l$ for some $l \in \mathbb{Z}$ and $c^2 = 4l^2$ so we have the contradiction that $4$ both divides $c^2$ and does not divide $c^2$.
Alternative proof

Assume \( a^2 + b^2 = c^2 \) then

\[ a^2 + b^2 \equiv c^2 \pmod{4}, \text{ in } \mathbb{Z}_4 \]

\[ 0^2 = 0, \quad 1^2 = 1, \quad 2^2 = 0, \quad 3^2 = 1 \]

Therefore \( c^2 \equiv 0 \) or \( c^2 \equiv 1 \pmod{4} \)

An odd number is either congruous to 1 or 3 \( \pmod{4} \) so if both \( a \) and \( b \) were odd we would have \( a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{4} \) not 0 or 1, so \( a \) and \( b \) cannot both be odd.
Problem 2  (12 points) Find all integer solutions of \(350x \equiv 210 \mod 140\)

\[
gcd(350, 140) = 70
\]

70 div 210 so we have solutions

we can cancel to end consider

the congruence \(5x \equiv 3 \mod 2\)

which is equivalent to \(x \equiv 1 \mod 2\)

so all solutions are

\[x = 1 + 2k \quad k \in \mathbb{Z}\]

the odd numbers.
Problem (12 points) Given the set \( A = \{ x \in \mathbb{Z} \mid 4 \text{ div } x \} \) prove that \( A \) is denumerable.

We know \( \mathbb{Z} \) is denumerable therefore there is a bijection \( f : \mathbb{N} \to \mathbb{Z} \). If we can find a bijection \( g : \mathbb{Z} \to A \) then \( g \circ f : \mathbb{N} \to A \) is a bijection from \( \mathbb{N} \) to \( A \).

Let \( g : \mathbb{Z} \to A \) be defined by \( g(z) = 4z \).

Then \( g \) is injective because \( z_1 \neq z_2 \Rightarrow g(z_1) \neq g(z_2) \).

\( g \) is surjective because given \( y \in A \) there is \( k \in \mathbb{Z} \) s.t. \( y = 4k \) by definition of \( A \), and therefore \( y = f(k) \).

So \( g \) is a bijection.
Problem 6 (12 points) Compute $542 \cdot 11^{2000} + 1023 \cdot 777 \mod 3$.

$542 \equiv 2 \mod 3$

$11^{2000} = (11^2)^{1000} \equiv 1 \mod 3$ (by Fermat's Little Theorem)

$1023 \equiv 0 \mod 3$

Therefore $542 \cdot 11^{2000} + 1023 \cdot 777 \equiv \boxed{2} \mod 3$
Problem 5 (16 points) Given the function $f : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ defined by $f(x) = 3x$ (so for example $f(5) = 3$)

1. Is $f$ injective? Prove your answer.

No \hspace{1cm} f(0) = f(4)

2. Is $f$ surjective? Prove your answer.

No the linear congruence $3x \equiv 1 \text{ mod } 12$ has no solutions since $\gcd(3, 12) = 3$ and $3$ does not divide $1$ so there is no $x$ in $\mathbb{Z}_{12}$ so $f(x) = 3x \equiv 1$.
**Problem 6** (14 points) Prove that for all sets $A, B, C$ and $D$

$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

Suppose $s \in (A \times B) \cup (C \times D)$

then $s \in A \times B$ or $s \in C \times D$

If $s \in A \times B$ then $s$ is a pair $(x, y)$ with $x \in A$ and $y \in B$.

If $s \in C \times D$ then $s$ is a pair $(x, y)$ with $x \in C$ and $y \in D$. In either case $s$ is a pair $(x, y)$ with $x \in A \cup C$ and $y \in B \cup D$. Therefore $s \in (A \cup C) \times (B \cup D)$
Problem (14 points) Given the following recursive definition

\[ f(1) = 3 \]
\[ f(2) = 3 \]
\[ f(n + 1) = f(n) \cdot f(n - 1) \]

- Prove that \( f(n) \) is odd for all \( n \in \mathbb{N} \)
- Prove that \( f(n) = 3^n \) where \( u_n \) is the \( n \)th Fibonacci number.

\[ P(n) : \forall f(n) = 3^u_n \text{ and } f(n) \text{ is odd} \]

Base case: \( n = 1 \)
\[ u_1 = 1 \Rightarrow f(1) = 3^1 = 3 \]
\[ f(1) \text{ is odd} \]

Base case: \( n = 2 \)
\[ u_2 = 1 \Rightarrow f(2) = 3^1 = 3 \]
\[ f(2) \text{ is odd} \]

Induction step: Assume \( P(k-1) \) and \( P(k) \) for \( k \geq 2 \), then
\[ f(k+1) = f(k) \cdot f(k-1) \]

By assumption, the product of 2 odd numbers is odd.
\[ f(k+1) = 3^u_k \cdot 3^u_{k-1} = 3^{u_k + u_{k-1}} \]
\[ f(k+1) = 3^{u_{k+1}} \] by definition of Fibonacci's numbers