Problem 1 (10 points) Assume $P$, $Q$ and $R$ are statements. Is $\neg P \lor \neg Q \lor R$ equivalent to $P \Rightarrow (Q \Rightarrow R)$? Justify your answer.

We proved in class that $A \Rightarrow B$ is equivalent to $B \lor \neg A$. Therefore $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(Q \Rightarrow R) \lor \neg P$, which is equivalent to $(R \lor Q) \lor \neg P$ which is equivalent to $\neg P \lor \neg Q \lor R$.

Note: I could also here used truth tables.
Problem 2 (10 points) Prove that $\sum_{i=1}^{n} u_{2i} = u_{2n+1} - 1$, where $u_i$ is the $i^{th}$ Fibonacci number.

Let $P(n)$ stand for $\sum_{i=1}^{n} u_{2i} = u_{2n+1} - 1$.

Proof by induction on $n$:

Base case: if $n = 1$, $\sum_{i=1}^{1} u_{2i} = u_{2} = 1$;

$u_{2+1} - 1 = u_{3} - 1 = 2 - 1 = 1$. So $P(1)$ is true.

Induction step: assume $P(k)$, then

$\sum_{i=1}^{k+1} u_{2i} = \sum_{i=1}^{k} u_{2i} + u_{2(k+1)} = u_{2k+1} - 1 + u_{2(k+1)}$

(by induction hypothesis) $= u_{2k+1} + u_{2k+2} - 1$

$= u_{2k+3} - 1$ (by definition of Fibonacci numbers) $= u_{2(k+1)+1} - 1$. So $P(k+1)$ is true.
Problem 3 (10 points) Assume $A$ is a denumerable set. Let $B = \{x \in \mathbb{N} \mid x \leq 10\}$ Prove $A \cup B$ is denumerable. You may assume $A$ and $B$ are disjoint.

Since $A$ is denumerable, there must exist a bijection of $\mathbb{N} \rightarrow A$. Define

$$g(n) = \begin{cases} n & \text{if } n \leq 10 \\ f(n-10) & \text{if } n > 10 \end{cases}$$

$g$ is injective: assume $n_1 \neq n_2$ then

1) If $n_1 \leq 10$ and $n_2 \leq 10$ clearly $g(n_1) \neq g(n_2)$

2) If $n_1 > 10$ and $n_2 > 10$ then clearly $g(n_1) \neq g(n_2)$

3) If $n_1 > 10$ and $n_2 \leq 10$ (or vice versa) then $g(n_1) \in A$ and $g(n_2) \in B$ so $g(n_1) \neq g(n_2)$ since $A \cap B = \emptyset$
$g$ is surjective: given $y \in A \cup B$,
if $y \in B$ then take $n = y$ and $g(n) = y$.
if $y \in A$ there is $m \in \mathbb{N}$ s.t.
$f(m) = y$, because $f$ is surjective.
Take $n = m + 10$ then $g(n) = f(m) = y$.
Problem 4 (10 points) Prove that no natural number of the form $4n+3$ (where $n \in \mathbb{N}$) is the sum of two squares (that is the sum of the squares of two natural numbers).

By contradiction assume

$4n+3 = m^2 + k^2$ for some $n, m, k \in \mathbb{N}$

Then we must have $3 \equiv m^2 + k^2 \pmod{4}$

But $0^2 \equiv 0 \pmod{4}$, $1^2 \equiv 1 \pmod{4}$, $2^2 \equiv 0 \pmod{4}$, $3^2 \equiv 1 \pmod{4}$

so the sum of two squares can be congruent to $0, 1 \pmod{4}$

but not $3$, and we have reached a contradiction.
Problem 5 (10 points) Prove that for any sets A, B and C
\[ A \times (B \cap C) = (A \times B) \cap (A \times C). \]

First I will prove \( A \times (B \cap C) \subseteq (A \times B) \cap (A \times C) \).
Assume \( x \in A \times (B \cap C) \) then \( x \) must be of the form \( x = (a, d) \) with \( a \in A \) and \( d \in B \cap C \).

This means \( d \in B \) and \( d \in C \) Therefore \( (a, d) \in A \times B \) and \( (a, d) \in A \times C \) therefore \( (a, d) \in (A \times B) \cap (A \times C) \).

Then I will prove \( (A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \).
Assume \( x \in (A \times B) \cap (A \times C) \) then \( x = (a, d) \) with \( a \in A \) and \( d \in B \) since \( x \in A \times B \) and also \( d \in C \) since \( x \in A \times C \). Therefore \( d \in B \cap C \) and \( x \in A \times (B \cap C) \).
Problem 6 (5 points) Find all integers $x$ that satisfy $12x \equiv 24 \pmod{30}$.

We can cancel 12:

$$x \equiv 2 \pmod{\frac{30}{\gcd(30,12)}} = 5$$

so $x = 2 + 5k$

(5 points) Compute $3^{400} + 150 \cdot 12^{27} \pmod{11}$

$$(3^{10})^{40} + 150 \cdot (1)^{27} \equiv 1 + 151 \equiv 8 \pmod{11}$$
Problem 7 (10 points) List all invertible elements of $\mathbb{Z}_{18}$.

$a$ is invertible in $\mathbb{Z}_{18}$ iff $\gcd(a, 18) = 1$. Therefore the invertible elements are $1, 5, 7, 11, 13, 17$. 