Problem 1 (10 points) Circle all the statements below that are equivalent to the negation of the statement "there is an integer $x$ such that either $x$ is prime or $x$ satisfies property 1 and property 2". No justification needed.

1. There is an integer $x$ such that $x$ is not prime and either $x$ does not satisfy property 1 or $x$ does not satisfy property 2.
2. For all integers $x$, $x$ is not prime and it does not satisfy property 1 and property 2.
3. For all integers $x$, if $x$ satisfies property 1 and 2 then $x$ is prime.
4. There is an integer $x$, $x$ is prime and $x$ satisfies property 1 or property 2.
5. For all integers $x$, either $x$ is not prime or $x$ does not satisfy property 1 or $x$ does not satisfy property 2.
6. For all integers $x$, $x$ is not prime and either $x$ does not satisfy property 1 or $x$ does not satisfy property 2.
7. For all integers $x$, if $x$ is prime, then $x$ does not satisfy property 1 and $x$ does not satisfy property 2.

The original statement is

$$\exists x \in \mathbb{Z} \text{ prime}(x) \lor (\text{P1}(x) \land \text{P2}(x))$$

The negation is:

$$\forall x \in \mathbb{Z} \neg \text{prime}(x) \lor (\neg \text{P1}(x) \lor \neg \text{P2}(x))$$

equivalent to

$$\forall x \in \mathbb{Z} \neg \text{prime}(x) \lor (\neg \text{P1}(x) \lor \neg \text{P2}(x)) \lor \text{P1}(x) \lor \text{P2}(x)$$

This is equivalent to 6 and to no other statement in the list above.
Problem 2 Define a function \( f : \mathbb{Z} \to \mathbb{Z} \) by:

\[
f(x) = \begin{cases} 
  x + 3 & \text{if } x \text{ is even} \\
  x + 1 & \text{if } x \text{ is odd}
\end{cases}
\]

Note \( x + 3 \) is odd, not \( x + 1 \) is even.

a) (5 points) Is \( f \) injective? (Give a proof).

Yes. We need to prove: \( \forall x_1, x_2 \in \mathbb{Z}, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \).

Assume \( x_1 \neq x_2 \) in \( \mathbb{Z} \) and consider 3 cases:

1) \( x_1, x_2 \) even: clearly \( x_1 + 3 \neq x_2 + 3 \).
2) \( x_1, x_2 \) odd: clearly \( x_1 + 1 \neq x_2 + 1 \).
3) One of \( x_1, x_2 \) even and the other odd: then one of \( f(x_1) \) \( f(x_2) \) is odd and the other even so \( f(x_1) \neq f(x_2) \).

b) (5 points) Is \( f \) surjective? (Give a proof).

Yes. We need to prove \( \forall y \in \mathbb{Z}, \exists x \in \mathbb{Z} \) \( f(x) = y \).

If \( y \) is odd solve \( y = x + 3 \) and take \( x = y - 3 \);

If \( y \) is even solve \( y = x + 1 \) and take \( x = y - 1 \);

If \( y \) is odd solve \( y = x + 3 \) and take \( x = y - 3 \);

If \( y \) is even solve \( y = x + 1 \) and take \( x = y - 1 \).
c)(10 points) Given $x$ in $\mathbb{Z}$, prove that $f^{2n}(x) = x + 4n$ for all $n \in \mathbb{N}$, where $f^{2n}$ means $f$ composed with itself $2n$ times, in other words

\[ f^{2n} = \underbrace{f \circ f \circ \cdots \circ f}_{2n \text{ times}} \]

By induction on $n$:

1) **Base case, $n = 1$**: If $x$ is even then $f^2(x) =$

\[ = f(f(x)) = f(x+3) = x+3+1 \quad (\text{since } x+3 \text{ is odd}) \]

\[ = x+4. \quad \text{If } x \text{ is odd then } f^2(x) = f(f(x)) \]

\[ = f(x+1) = x+1+3 \quad (\text{since } x+1 \text{ is even}) = x+4 \]

so $\forall x \in \mathbb{Z}$, $f^2(x) = x+4$

2) **Induction step**: Assume $f^{2k}(x) = x + 4k$

Then $f^{2(k+1)}(x) = f^{2k+2}(x) = f^{2k} \circ f^{2k}(x) =$

\[ = f^2(x+4k) \quad (\text{by induction assumption}) \]

\[ = x + 4(k+1) \quad (\text{by 1}) \quad = x + 4(k+1) \]
**Problem 3 (10 points)** Prove that, for all integers $x$,
10 divides $x$ if and only if 2 divides $x$ and 5 divides $x$. (You can assume that the product of two odd numbers is odd without having to prove this).

$$10 \div x \Rightarrow 2 \div x \land 5 \div x \quad \text{Assume } 10 \div x.$$  

Then $x = 10k$ for some $k \in \mathbb{Z}$, so $x = 2(5k)$ and $5k \in \mathbb{Z}$, therefore $2 \div x$ and $x = 5(2k)$ and $2k \in \mathbb{Z}$ so $5 \div x$.

$$2 \div x \land 5 \div x \Rightarrow 10 \div x.$$ Assume $2 \div x \land 5 \div x$, then $x = 2m$ for some $m \in \mathbb{Z}$ and $x = 5n$ for some $n \in \mathbb{Z}$. Since $x$ is even, $5n$ must be even and therefore $n$ must be even; since 5 is odd and the product of two odd integers is odd, therefore $n = 2k$ for some $k \in \mathbb{Z}$ and substituting in $(**)$ we get $x = 5(2k) = 10k$, therefore $10 \div x$.  

5
Problem 4 (10 points) Let $A$, $B$ and $C$ be non empty sets. Prove that 
$(A \subseteq B) \land (A \cap C = \emptyset) \Rightarrow P(A) \subseteq P(B - C)$

Assume $A \subseteq B$ and $A \cap C = \emptyset$. We need to show that 
$\forall S \in P(A) \Rightarrow S \in P(B - C)$.

Assume $S \in P(A)$, this means $S \subseteq A$ by the definition of power set; now if $x \in S$ then $x \in A$ and $x \in B$, since $A \subseteq B$ and $x \notin C$ since $A \cap C = \emptyset$, so $x \in B - C$; therefore $S \subseteq B - C$ and this means $S \in P(B - C)$. 