1. The negation of 
\[ \forall x \in \mathbb{R} \left( \left( 2d \uparrow x \lor 3d \downarrow x \right) \right) \]
is \[ \exists x \in \mathbb{R} \left( 7 \uparrow 2d \downarrow x \lor 7 \uparrow 3d \downarrow x \right) \]
The only statement equivalent to this is (e)

2. f is injective. Proof:

Given \( x_1, x_2 \in \mathbb{R}, x_1 \neq x_2 \) clearly

\( f(x_1) \neq f(x_2) \) if both \( x_1, x_2 \)

are \( \geq 0 \) or both \( x_1, x_2 \) are \( < 0 \)

If one of \( x_1, x_2 \) is non-negative, say \( x_1 \)

and the other negative, say \( x_2 \).

Then \( f(x_1) \) is non-negative and

\( f(x_2) \) is negative, so \( f(x_1) \neq f(x_2) \)
$f$ is not surjective. For example, given $1 \in \mathbb{Z}$, there is no $x \in \mathbb{Z}$ such that $f(x) = 1$, since $f(x)$ is either divisible by 2 or 3 and 1 is not divisible by 2 or 3.

$-1$

$g$ is not injective. Proof:

$g(3) = g(-1)$.

$g$ is surjective. Proof: We need to prove $\forall y \in \mathbb{N} \exists x \in \mathbb{Z}$ such that $f(x) = y$. Given $y$, take $x = y$.

Then $x \geq 0$ and $f(x) = y$. 


3. We will prove

1) \( A \times B = B \times A \Rightarrow A = B \)

2) \( A = B \Rightarrow A \times B = B \times A \)

Proof of 1) By contradiction.
Assume \( A \times B = B \times A \) and \( A \neq B \).

Then there must be some
\( x, \ x \in A \) and \( x \notin B \)
(or, vice versa and then the proof is completely analogous)

Let \( y \in B \) be some element in \( B \), which

must exist, since \( B \neq \emptyset \)
Then \((x, y) \in A \times B\) but \((x, y) \notin B \times A\)

so \(A \times B \neq B \times A\) which contradicts

the assumption \(A \times B = B \times A\).

2) Obvious

\((A \cup B) \subseteq (B \cap C) \Rightarrow A \subseteq C\)

Proof: Assume \(A \cup B \subseteq B \cap C\)

and \(x \in A\); we need to show \(x \in C\).

since \(x \in A\) then \(x \in A \cup B\) and therefore

\(x \in B \cap C\) so \(x \in C\).

\((A \cap B) \subseteq P(A) \cap P(B)\)

Proof assume \(S \subseteq P(A \cap B)\)

Then \(S \subseteq A \cap B\). Let \(x \in S\)

then \(x \in A \cap B\) so \(x \in A\) and \(x \in B\)
we have shown,
so \( \forall x \in S \Rightarrow x \in A \land x \in B \)
so \( S \subseteq A \) and \( S \subseteq B \),

Therefore \( S \in P(A) \) and \( S \in P(B) \) so
\( S \in P(A) \cap P(B) \).

Therefore
\( S \in P(A \cap B) \Rightarrow S \in P(A) \cap P(B) \)

so \( P(A \cap B) \subseteq P(A) \cap P(B) \)

(6) Proof by strong induction:

Base case: the game starts with two
stocks of 1 coin each. Player 1 must remove one coin from
one stock. Player 2 removes a
coin from the other stock and wins.

Induction step: assume player 2 can win
the games that start with stacks of
\( 2, 3, \ldots, k \) coins.

Consider the game that starts with

2 stacks of \( k+1 \) coins. If player
1 removes all coins from one stock
Then player 2 removes all coins from the other stack and wins. If player 1 removes \( m < k+1 \) coins from one stack then player 2 removes \( m \) coins from the other stack and then the players play the game with 2 stacks of \( k+1-m \) coins and, by induction assumption, player 2 can win this game.

By induction

Base case: when \( n = 1 \) \( \sum_{i=1}^{1} c_i^3 = 1 \)

\[ \text{and} \quad \left( \frac{1((1+1)^2)}{2} \right) = 1 \]

Induction step: assume \( c_i = 1 \) for all \( i > 1 \)
\[
\sum_{i=1}^{k} i^3 = \frac{(k(k+1))^2}{2} + \eta
\]

\[
\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \\
(\frac{k(k+1)}{2})^2 + (k+1)^3 = \\
(\frac{k+1}{\zeta})^2 \cdot \left( k^2 + \zeta(k+1) \right) = \\
(\frac{k+1}{\zeta})^2 \cdot (k+2)^2 = \left( \frac{(k+1)(k+2)}{2} \right)^2
\]
(8) Proof by induction

Base case: for \( n = 1 \) there are 2 strings of length 1 and neither of them contains 2 consecutive ones.

\( U_3 = 2 \) so the statement is true for \( n = 1 \).

For \( n = 2 \) (you will need to look at the induction step to see why I need to look at \( n = 2 \) as well.)

There are four strings of length 2 namely

00 01 10 11
We need to eliminate
11 so we are left with
3 strings and \( u_4 = u_3 + u_2 = 3 \)
so the statement is true
for \( n = 2 \).

Induction step: assume
the statement is true
for \( k \) and \( k-1 \). We
need to show the statement
is true for \( k+1 \). Consider
a string of length \( k+1 \).

\[ \overline{11111} \text{ if the string} \]
starts with \( 0 0 \). \[ \begin{array}{c}
\text{length } k
\end{array} \]
then the rest of the string

can be filled in \( U_{k+2} \) ways

by our induction assumption.

If the string starts with

a 1 the next element of

the string must be a 0

and the rest of the string

\[ \text{can be filled} \]

\[ \text{length } k-1 \]

in \( U_{(k-1)+2} \) ways by our

induction assumption.

Therefore, in total we

have \( U_{k+1} + U_{k+2} = U_{k+3} = \]

\( U_{(k+1)+2} \) strings.
9) \[ \begin{array}{c|c}
 n & \alpha_n \\
 4 & 2 \\
 5 & 2^2 \\
 6 & 2^3 \\
 7 & 2^6 \\
 8 & 2^{11} \\
 9 & 2^{20} \\
 10 & 2^{37} \\
 11 & 2^{69} \\
\end{array} \]

So certainly \( \alpha_n \geq 2^{2n} \)

for \( n = 9, 10, 11 \)

Induction step: assume

\( \alpha_k \geq 2^k \) and \( \alpha_{k-1} \geq 2^{2(k-1)} \)

and \( \alpha_{k-2} \geq 2^{2(k-2)} \) Then

\[ \alpha_{k+1} \geq 2^k \cdot 2^{2(k-1)} \cdot 2^{2(k-2)} = 2^{6k-6} \]

If \( k \geq 9 \) then \( 6k-6 \geq 2^{(k+1)} \)

because by algebra \( (x) \) is
equivalent to
\[ 4k \geq 8 \quad \text{end to equivalent} \]
\[ k \geq 2 \quad \text{which is true} \]
Therefore \[ q^{k+1} \geq 2^{6k-6} \geq 2^{2(k+1)} \]