2. Calculate the following integrals.

(a) (10 points) [redacted] $\int_1^2 x e^{\sqrt{x}}\,dx$

$$v' = x \qquad v = \frac{x^2}{2} \qquad u = e^{\sqrt{x}}$$

$$v u' = \frac{x^2}{2}\,\frac{1}{2\sqrt{x}}\,e^{\sqrt{x}}$$

Integration by parts does not work

$$u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}\,dx \qquad \int x\cdot e^u\, 2\sqrt{x}\, du, \quad 2\int_1^{\sqrt{2}} \underbrace{u^3}_{a} \underbrace{e^u}_{b'}\, du = 2u^3e^u\Big|_1^{\sqrt{2}} - 2\cdot 3\int_1^{\sqrt{2}} u^2 e^u\, du$$

$$= 2u^3e^u\Big|_1^{\sqrt{2}} - 6u^2e^u\Big|_1^{\sqrt{2}} + 6\cdot 2\int_1^{\sqrt{2}} u e^u\, du = 2u^3e^u - 6u^2e^u\Big|_1^{\sqrt{2}} + 12ue^u\Big|_1^{\sqrt{2}} - 12\int_1^{\sqrt{2}} e^u du$$

$$2u^3e^u - 6u^2e^u + 12ue^u - 12e^u\Big|_1^{\sqrt{2}} = (16\sqrt{2} - 24)e^{\sqrt{2}} + 4e$$

Here we did a substitution and then repeated integration by parts

(b) (13 points) $\int_0^{\frac{\pi}{3}} \sin^3(x) + \sec x \cdot \tan^3(x)\, dx$

$$\int_0^{\pi/3} \sin^3(x)\,dx = \int_0^{\pi/3} \sin^2(x)\cdot \underbrace{\sin x\, dx}_{\text{almost derivative of } \cos x} \qquad u = \cos x, \quad du = -\sin x\, dx$$

$$-\int_{\cos(0)}^{\cos(\pi/3)} (1-u^2)\,du = \int_{1/2}^{1} (1-u^2)\,du = u - \frac{u^3}{3}\Big|_{1/2}^{1} = \frac{5}{24}$$

$$\int_0^{\pi/3} \sec x \tan^3 x\, dx = \int_0^{\pi/3} \underbrace{\sec x \tan x}_{\text{der of } \sec(x)}\ \underbrace{\tan^2 x}_{= \sec^2 x - 1}\, dx \qquad u = \sec x, \quad du = \sec x \tan x\, dx$$

$$\int_{\sec 0}^{\sec \pi/3} u^2 - 1\, du = \frac{u^3}{3} - u\Big|_1^2 = \frac{4}{3}$$

$$\text{so} \quad \int_0^{\pi/3} \sin^3(x) + \sec x \tan^3 x\, dx = \frac{5}{24} + \frac{4}{3} = \frac{37}{24}$$

2. (12 total points) Evaluate the following indefinite integrals.

(a) (6 points) $\displaystyle\int \frac{x+2}{\sqrt{x^2+2x-3}}\,dx$

1) Complete the square : $x^2+2x-3=(x+1)^2-4$

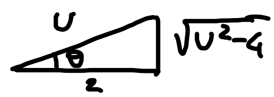
2) Do a substitution $\displaystyle\int \frac{x+2}{\sqrt{(x+1)^2-4}}\,dx$ $\quad u=x+1$, $du=dx$ $\quad \displaystyle\int \frac{u+1}{\sqrt{u^2-4}}\,du$

3) Inverse trig sub $\quad u=2\sec\theta$, $du=2\sec\theta\tan\theta$

$$\int \frac{2\sec\theta+1}{\underbrace{\sqrt{4(\sec^2\theta-1)}}_{\cancel{2\tan\theta}}}\,\cancel{2}\sec\theta\cancel{\tan\theta}\,d\theta = \int 2\sec^2\theta+\sec\theta\,d\theta$$

$= 2\tan\theta + \ln|\tan\theta+\sec\theta| + C$

4) Go back to x



$\tan\theta = \dfrac{\sqrt{u^2-4}}{2}$, $\sec\theta = \dfrac{u}{2}$

$$\cancel{2}\frac{\sqrt{(x+1)^2-4}}{\cancel{2}} + \ln\left|\frac{\sqrt{(x+1)^2-4}}{2} + \frac{(x+1)}{2}\right| + C$$

Use triangle do not write something like $\tan(\sec^{-1}(\frac{x+1}{2}))$

(b) (6 points) $\displaystyle\int \frac{x+1}{x^3+x^2-6x}\,dx$

$x(x^2+x-6) = x(x+3)(x-2)$, $\quad \dfrac{x+1}{x^3+x^2-6x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2} = \dfrac{A(x+3)(x-2)+Bx(x-2)+Cx(x+3)}{x(x+3)(x-2)}$

$A(x+3)(x-2)+Bx(x-2)+Cx(x+3) = x+1$

$x=0 \quad -6A=1 \quad A=-1/6$

$x=-3 \quad 15B=-2 \quad B=-2/15$

$x=2 \quad 10C=3 \quad C=\frac{3}{10}$

$$\int \frac{-1/6}{x} + \frac{-2/15}{x+3} + \frac{\frac{3}{10}}{x-2}\,dx = -\frac{1}{6}\ln|x| - \frac{2}{15}\ln|x+3| + \frac{3}{10}\ln|x-2| + C$$

4. This problem has three parts. In all three parts a 10 feet chain is dangling from the roof of a tall building.

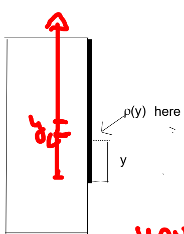
   (a) (10 points) The chain weights 2 lb/ft. How much work is needed to pull it up to the top of the building?.

$$W = \int_0^{10} 2x\,dx = x^2 \Big|_0^{10} = 100 \text{ ft-lb}$$

b) How much work is needed to pull up 3 feet and leave 7 ft dangling?

$$\int_0^3 2x\,dx + \underbrace{2\cdot 7}_{F}\cdot\underbrace{3}_{d} = x^2\Big|_0^3 + 42 = 51 \text{ ft-lb}$$

   (c) (7 points)The chain is made of different materials. Its weight density is not an uniform 2 lb/ft as in part (a) but it is given by $\rho(y) = \frac{11-y}{10}$ lb/ft where y is distance from the bottom of the chain. Set up an integral that calculates how much work is needed to pull the chain up to the top of the building.Do not evaluate the integral.

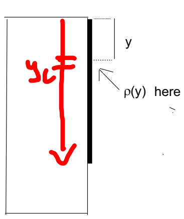


$$W_c = \rho(y_c)\,\Delta y \cdot (10-y)$$

$$W = \int_0^{10} \frac{11-y}{10}\,(10-y)\,dy$$

you could also write $\int_0^{10} \frac{11-(10-y)}{10}\cdot y \cdot dy$

   (d) (7 points)The chain is made of different materials. Its weight density is given by $\rho(y) = \frac{11-y}{10}$ lb/ft where y is distance from the top of the chain.Set up an integral that calculates how much work is needed to pull the chain up to the top of the building.Do not evaluate the integral.

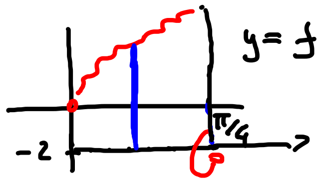


$$W_c = \rho(y_c)\,\Delta y \cdot y$$

$$W = \int_0^{10} \frac{11-y}{10}\cdot y\,dy$$

5. **[12 points]** Let $\mathcal{R}$ be the region in the x-y plane below $y = \sec(x)\tan(x)$ and above $y = -2$ from $x = 0$ to $x = \dfrac{\pi}{4}$.

(a) Write an integral to compute the volume of the solid formed by revolving $\mathcal{R}$ around the line $y = -2$.



$V_{slice} = \pi r^2 \Delta x$

$r = (\sec x \tan x + 2)$

$$V = \pi \int_0^{\pi/4} (\sec x \tan x + 2)^2 \, dx$$

(b) Evaluate the integral from part (a).

$$\pi \int_0^{\pi/4} \sec^2 x \tan^2 x + 4 \sec x \tan x + 4 \, dx$$

$$\int_0^{\pi/4} \sec^2 x \tan^2 x \, dx \qquad u = \tan x \quad du = \sec^2 x\,dx \qquad \int_{\tan 0}^{\tan \frac{\pi}{4}} u^2 = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\int_0^{\pi/4} \sec x \tan x \, dx = \sec x \Big|_0^{\pi/4} = \sqrt{2} - 1$$

$$\int_0^{\pi/4} 4 \, dx = 4x \Big|_0^{\pi/4} = \pi$$

so $V = \pi \left(\frac{1}{3} + 4(\sqrt{2}-1) + \pi\right) = \pi\left(4\sqrt{2} + \pi - \frac{11}{3}\right)$

3.

(a) [**3 points**] Write an integral expression to compute the average value of $f(x) = \cos(x^2)$ over the interval $[-1, 2]$. DO NOT try to compute the integral.

$$\frac{1}{2-(-1)} \int_{-1}^{2} \cos(x^2)\,dx = \frac{1}{3}\int_{-1}^{2} \cos(x^2)\,dx$$

(b) [**5 points**] Use the Trapezoidal Rule with $n = 6$ subintervals to approximate the integral from part (a).

Your answer should either be in exact form (but simplify all you can), or in decimal form with at least 4 digits of precision.

$$\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$$

-1	$-1/2$	0	$1/2$	1	$3/2$	2
x_0	x_1	x_2	x_3	x_4	x_5	x_6

$$Av \approx \frac{1}{3}\cdot\frac{1}{4}\left(\cos 1 + 2\cos\frac{1}{4} + 2\cos 0 + 2\cos\frac{1}{4} + 2\cos 1 + 2\cos\left(\frac{9}{4}\right) + \cos(4)\right)$$

$$\approx 0.6655$$

Use Simpson with $n = 4$:

$$\Delta x = \frac{3}{4}$$

$-1 \quad -\frac{1}{4} \quad \frac{2}{4} \quad \frac{5}{4} \quad 2$

$$S_4 = \frac{3}{4\cdot 3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

$$Av \approx \frac{1}{3}\cdot\frac{1}{4}\left(\cos 1 + 4\cos\frac{1}{16} + 2\cos\left(\frac{1}{4}\right) + 4\cos\left(\frac{25}{16}\right) + \cos 4\right)$$

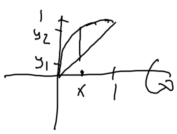
0.487

Set up , but do not calculate, integrals that give the volume of the solid obtained rotating the region in the first quadrant bound by the curves $y=\sqrt{x}$ and $y=x$. Find intersection : $x=\sqrt{x}$

$x^2=x$ if $x=0$ or $x=1$

$y=0$ $y=1$

1. Around the x axis

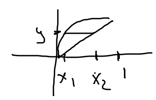


$\pi(r_1^2-r_2^2)\,dx$

$r_1=y_2=\sqrt{x}$

$r_2=y_1=x$

$\boxed{\int_0^1 \pi(x-x^2)\,dx}$ **or**



$2\pi r h\,\Delta y$

$r=y$

$h=x_2-x_1$

$h=y-y^2$

$\boxed{2\pi\int_0^1 y(y-y^2)\,dy}$

2. Around the y axis

$2\pi r h\,dx$

$r=x$

$h=y_2-y_1=\sqrt{x}-x$

$\boxed{\int_0^1 2\pi x(\sqrt{x}-x)\,dx}$ or

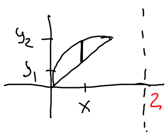
$\pi(r_1^2-r_2^2)\,dy$

$r_1=x_2=y$

$r_2=x_1=y^2$

$\boxed{\pi\int_0^1 (y^2-y^4)\,dy}$

3. Around the axis $x=2$

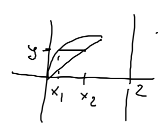


$2\pi r h\,\Delta x$

$r=2-x$

$h=y_2-y_1=\sqrt{x}-x$

$\boxed{\int_0^1 2\pi(2-x)(\sqrt{x}-x)\,dx}$ or



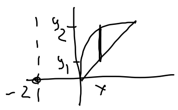
$\pi(r_1^2-r_2)^2\,\Delta y$

$r_1=2-x_1=2-y^2$

$r_2=2-x_2=2-y$

$\boxed{\pi\int_0^1 (2-y^2)^2-(2-y)^2\,dy}$

4. Around the axis $x=-2$

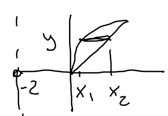


$2\pi r h\,\Delta x$

$r=2+x$

$h=y_2-y_1=\sqrt{x}-x$

$\boxed{\int_0^1 2\pi(2+x)(\sqrt{x}-x)}$ or



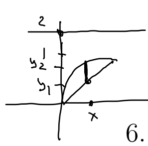
$\pi(r_1^2-r_2^2)\,\Delta y$

$r_1=2+x_2=2+y$

$r_2=2+x_1=2+y^2$

$\boxed{\pi\int_0^1 (2+y)^2+(2+y^2)^2\,dy}$

5. Around the axis $y=2$

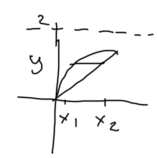


$\pi(r_1^2-r_2^2)\,\Delta x$

$r_1=2-y_1=2-x$

$r_2=2-y_2=2-\sqrt{x}$

$\boxed{\int_0^1 \pi((2-x)^2-(2-\sqrt{x})^2)\,dx}$ or



$V=2\pi r h\,\Delta y$

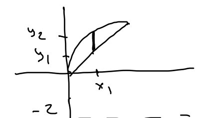
$r=2-y$

$h=x_2-x_1$

$h=y-y^2$

$\boxed{2\pi\int_0^1 (2-y)(y-y^2)\,dy}$

6. Around the axis $y=-2$

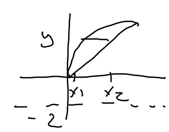


$\pi(r_1^2-r_2^2)\,\Delta X$

$r_1=2+y_2=2+\sqrt{x}$

$r_2=2+y_1=2+x$

$\boxed{\int_0^1 \pi[(2+\sqrt{x})^2-(2+x)^2]\,dx}$ or



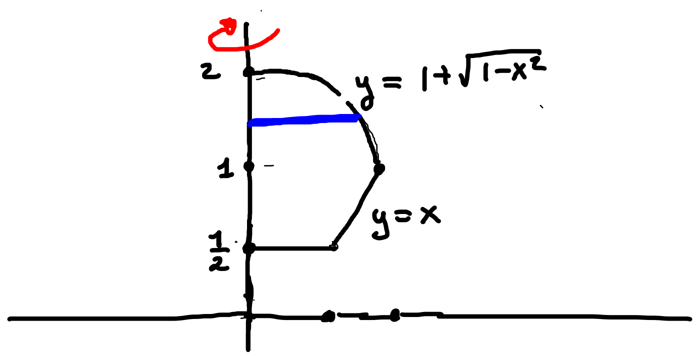
$V=2\pi r h\,\Delta y$

$r=2+y$

$h=x_2-x_1=y-y^2$

$\boxed{2\pi\int_0^1 (2+y)(y-y^2)\,dy}$

Set up but do not calculate integral(s) that give the volume of the solid obtained by rotating the region below around the y axis

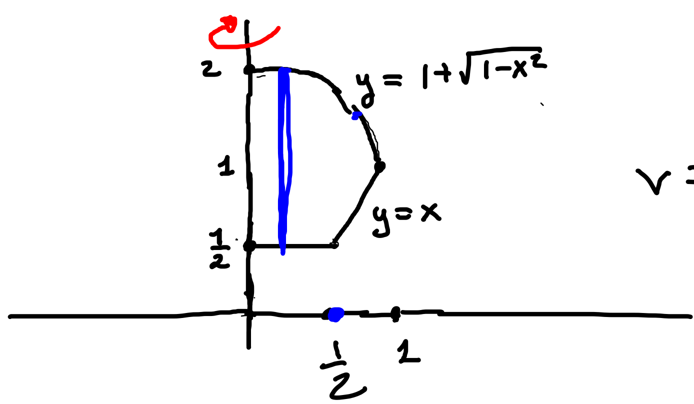


$$V_{slice} = \pi r^2 \Delta y$$

$$r = x \;,\; r^2 = x^2 = \begin{cases} y^2 & \text{if } \frac{1}{2} \le y \le 1 \\ 1-(y-1)^2 & \text{if } 1 \le y \le 2 \end{cases}$$

$$V = \pi \int_{1/2}^{1} y^2\,dy + \pi \int_{1}^{2} 1-(y-1)^2\,dy$$

or



$$V = 2\pi r h \,\Delta x$$

$$r = x$$

$$h = \begin{cases} 1+\sqrt{1-x^2}-\frac{1}{2} & \text{if } 0 \le x \le \frac{1}{2} \\ 1+\sqrt{1-x^2}-x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$V = 2\pi \int_0^{1/2} x\left(1+\sqrt{1-x^2}-\frac{1}{2}\right)dx + 2\pi \int_{1/2}^{1} x\left(1+\sqrt{1-x^2}-x\right)dx$$