1. **[5 points per part]** Compute each limit. You may use any techniques you know.

   If a limit does not exist or is infinite, say so, and explain.

   (a) $\lim_{x\to 8} \frac{\sqrt{x-4}+2}{x-3}$, $\lim_{x\to 8} \frac{\sqrt{x-4}+2}{x-8}$, $\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8}$

$\lim_{x\to +\infty} \frac{\sqrt{x-4}+2}{x-3}$

Solutions: see next page

   (b) $\cancel{\lim_{t\to 0} \frac{\sin(at) - bt + ct^2}{t}}$

   (c) $\lim_{x\to\infty} \sin\left(\frac{\overbrace{\pi x + 6}^{\to \pi/4}}{\sqrt{4x^2+2x}+2x}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

First calculate $\lim_{x\to+\infty} \frac{\pi x + 6}{\sqrt{4x^2+2x}+2x} = \frac{\pi}{4}$ since $\frac{\pi x}{\sqrt{4x^2}+2x} = \frac{\pi \cancel{x}}{4\cancel{x}}$, for $x>0$

Leading terms

or $\frac{\cancel{x}\left(\pi + \frac{6}{x}\right)}{2\cancel{x}\left(\sqrt{1+\frac{2x}{4x^2}}+1\right)} \longrightarrow \frac{\pi}{2\cdot 2} = \frac{\pi}{4}$ ($\frac{6}{x} \to 0$, $\frac{2x}{4x^2} \to 0$)

a) i) $\lim_{x \to 8} \frac{\sqrt{x-4} + 2}{x-3} = \frac{4}{5}$ since if you plug in $x = 8$: you get $\frac{4}{5}$

a) ii) $\lim_{x \to 8} \frac{\sqrt{x-4} + 2}{x-8} = DNE$:

plug in $x=8$ you get $\frac{4}{0}$, $\left(\frac{\text{constant} \neq 0}{0}\right)$

Consider $\lim_{x \to 8^+} \frac{\overbrace{\sqrt{x-4} + 2}^{\to 2}}{\underbrace{x-8}_{\to 0^+}} = +\infty$

$\lim_{x \to 8^-} \frac{\overbrace{\sqrt{x-4} + 2}^{\to 2}}{\underbrace{x-8}_{\to 0^-}} = -\infty$

The limits from the right and left are different

a) iii) $\lim_{x \to 8} \frac{\sqrt{x-4} - 2}{x-8} = \frac{1}{4}$

plug in $x = 8$ you get $\frac{0}{0}$ (both top and bottom are 0)

Try rationalizing $\frac{\sqrt{x-4} - 2}{x-8} \cdot \frac{\sqrt{x-4} + 2}{\sqrt{x-4} + 2} = \frac{x - \cancel{4} - 4}{\cancel{x-8}} \cdot \frac{1}{\sqrt{x-4} + 2}$

$\lim_{x \to 8} \frac{\sqrt{x+4} - 2}{x-8} = \lim_{x \to 8} \frac{1}{\sqrt{x-4} + 2} = \frac{1}{4}$

a) iv) $\lim_{x \to +\infty} \frac{\sqrt{x-4} + 2}{x-3} = 0$

$\to$ leading term

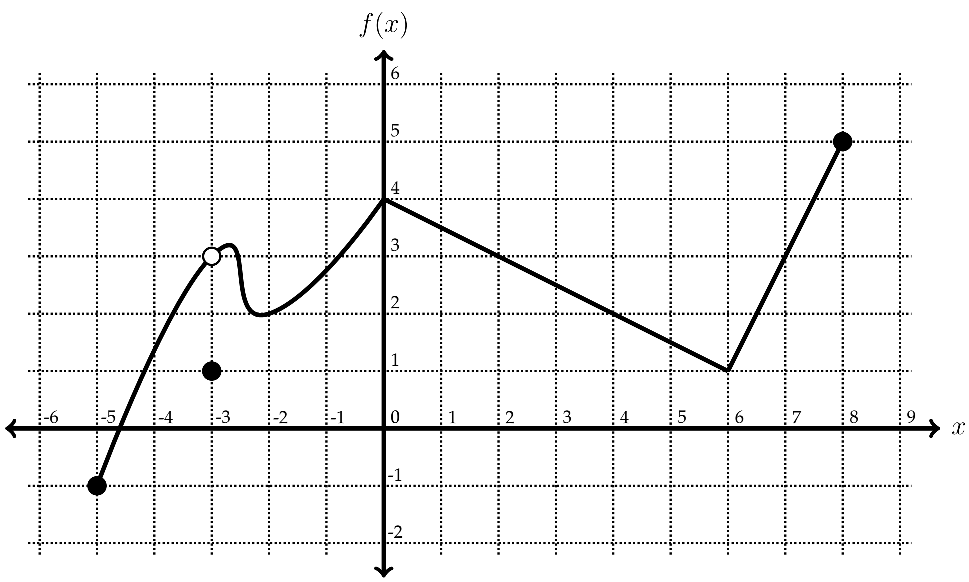
This much work would be ok in a test, but see next page for extra explanation

$$\frac{\sqrt{x-4}+2}{x-3} = \frac{\sqrt{x\left(1-\frac{4}{x}\right)}+2}{x\left(1-\frac{3}{x}\right)} =$$

$$= \frac{\sqrt{x}\left(\sqrt{1-\frac{4}{x}}+\frac{2}{\sqrt{x}}\right)}{x\left(1-\frac{3}{x}\right)} = \underbrace{\frac{1}{\sqrt{x}}}_{\to 0} \cdot \frac{\sqrt{1-\underbrace{\frac{4}{x}}_{\to 0}}+\underbrace{\frac{2}{\sqrt{x}}}_{\to 0}}{1-\underbrace{\frac{3}{x}}_{\to 0}}$$

$$= 0 \cdot 1 = 0$$

5. The graph of $f(x)$ is shown below.



Cool graph, right? Use it to answer the following questions.

(a) **[3 points]** Compute $\lim_{x\to -3}[\underbrace{f(x)}_{3}\cdot \underbrace{f(\underbrace{x+1}_{-2})}_{2}]$. $= 6$

(b) **[3 points]** For what constant c does $\lim_{x\to 3}\frac{f(x)-c}{x-3}$ exist?

If $c = f(3)$ the limit exists (see c))

If $c \neq f(3)$ the limit is of the form $\frac{k \text{ (constant} \neq 0)}{0}$

and $\lim_{x\to 3^+}\frac{f(x)-c}{x-3} \to \frac{k}{0^+} \neq \lim_{x\to 3^-}\frac{f(x)-c}{x-3} \to \frac{k}{0^-}$ so the limit DNE

(c) **[3 points]** Compute the limit from part (b), using the value of c you chose.

Derivative $f'(3)$ = slope of tangent at $P(3, f(3))$

so $f'(3) = \frac{1-4}{6} = -\frac{1}{2}$

~~(d) **[4 points]** Let~~ $\cancel{g(x) = \frac{f'(x)}{f(x)}}$~~. What is~~ $\cancel{g'(7)}$~~?~~

3. (20 points) (a) (8 points) Algebraically simplify the expression inside the following limit:

$$\lim_{h \longrightarrow 0} \frac{\sqrt{(3+h)^2+16}-5}{h}.$$

(b) (5 points) Using part (a), find this limit.

(c) (7 points) This limit is the derivative of what function $f(x)$ at what point?

$$\frac{\sqrt{9+6h+h^2+16}-5}{h} \cdot \frac{\sqrt{9+16+6h+h^2}+5}{\sqrt{9+16+6h+h^2}+5} = \frac{25+6h+h^2-25}{h\left(\sqrt{25+6h+h^2}+5\right)}$$

$$= \frac{6+h}{\sqrt{25+6h+h^2}+5}$$

$$\lim_{h \to 0} \frac{\sqrt{(3+h)^2+16}-5}{h} = \lim_{h \to 0} \frac{6+h}{\sqrt{25+6h+h^2}+5} = \frac{6}{10} = \frac{3}{5}$$

c) We want to think of it as

$$\lim_{h=0} \frac{f(h+x_0)-f(x_0)}{h} = f'(x_0)$$

$f(x_0) = 5$

$f(h+x_0) = \sqrt{(3+h)^2+16}$ x

what is $f(x)$? $\sqrt{x^2+16}$

what is x_0? 3

$\boxed{f'(3)}$

(CONTINUED ON NEXT PAGE)

2. (12 total points) Find the following limits. In each case your answer should be either a number, $+\infty$, $-\infty$ or DNE. Please show your work.

(a) (4 points) $\lim_{t\to 2^-} \dfrac{t^2-4}{|t-2|}$ $\to 0$, $\to 0$

Do some algebra: $\dfrac{(t-2)(t+2)}{-(t-2)} = -(t+2)$ → you will probably need to explain this

$\lim_{t\to 2^-} \dfrac{t^2-4}{|t-2|} = \lim_{t\to 2^-} -(t+2) = -4$

Remind them indeterminate forms $\infty - \infty$ $\frac{\infty}{\infty}$ $\frac{0}{0}$ $0\cdot\infty$

(b) (4 points) $\lim_{x\to\infty} \left(x - \sqrt{x^2-10x}\right)$

$\infty - \infty$ indeterminate form

(In $\sqrt{x^2-10x}$ x^2 is dominant so $\lim_{x\to+\infty} \sqrt{x^2-10x} = +\infty$)

Try rationalizing $(x-\sqrt{x^2-10x})\dfrac{(x+\sqrt{x^2-10x})}{x+\sqrt{x^2-10x}} = \dfrac{x^2-(x^2-10x)}{x+\sqrt{x^2-10x}} = \dfrac{10x}{x+\sqrt{x^2-10x}}$

$\lim_{x\to+\infty} \dfrac{10x}{x+\sqrt{x^2-10x}} = \lim_{x\to+\infty} \dfrac{10\cancel{x}}{\cancel{x}(1+\sqrt{1-\frac{10}{x}})} = \dfrac{10}{2} = 5$

(or look at $\dfrac{10x}{x+\sqrt{x^2}} = \dfrac{10}{2}$ (for $x>0$)

(c) (4 points) $\lim_{x\to\infty} \dfrac{2x^2+3x\ln x+2^{-x}}{5x^2+9x\ln x+\pi\cdot 2^{-x}} = \dfrac{2}{5}$ ($2^{-x}\to 0$, $\to 0$)

Dominant terms circled

$\lim_{x\to\infty} \dfrac{\text{Top}}{\text{bottom}}$

1) Find the fastest growing term on top, say $f(x)$
2) Find the fastest growing term on bottom, say $g(x)$
3) Consider $\lim_{x\to+\infty} \dfrac{f(x)}{g(x)}$

Recall bounded $< \ln(x) < \sqrt{x} < x < \underset{n>1}{x^n} < e^x$

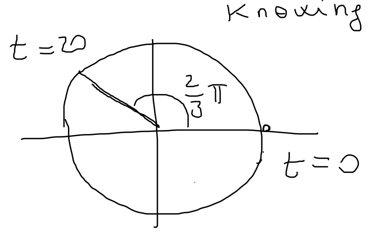
5. (25 points) A particle is traveling at constant angular velocity $\pi/3$ rad/sec counterclockwise around the circle of radius 2 centered at the origin. At time $t = 0$ it is at the point $(2, 0)$. At time $t = 20$ sec the particle flies off the circle and continues at constant velocity along the tangent line. NOTE: Your answers may involve π and square roots.

(a) (5 points) Give parametric equations for the motion of the particle for $0 \leq t \leq 20$.

(b) (10 points) At the instant when the particle flies off the circle find its x- and y-coordinates ~~and its horizontal velocity and vertical velocity.~~

(c) (10 points) Give parametric equations for the motion of the particle for $t \geq 20$.

Knowing the horizontal velocity V_x is $-\frac{\pi\sqrt{3}}{3}$ and the vertical velocity V_y is $-\frac{\pi}{3}$



a) $x(t) = 2\cos\left(\frac{\pi}{3}t\right)$

$y(t) = 2\sin\left(\frac{\pi}{3}t\right)$

b) $x(20) = 2\cos\left(\frac{\pi}{3}20\right) = 2\cos\left(6\pi + \frac{2}{3}\pi\right) = 2\left(-\frac{1}{2}\right) = -1$

$y(20) = 2\sin\left(6\pi + \frac{2}{3}\pi\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

c) $x = x_1 + V_x(t-20) = 1 - \frac{\pi\sqrt{3}}{3}(t-20)$

$y = y_1 + V_y(t-20) = \sqrt{3} - \frac{\pi}{3}(t-20)$