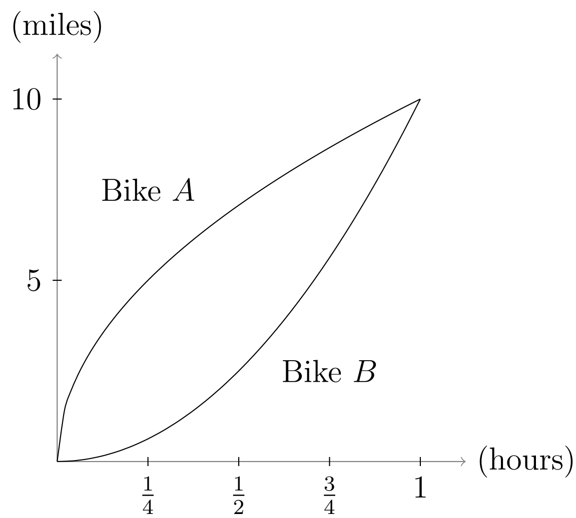
Worksheet for Week 2: Graphs and limits

In this worksheet, you'll practice using the graph of an object's position to learn about its velocity. You'll also learn a useful technique for computing limits of certain types of functions at points where the function might not be defined.

1. Consider the graph below, which shows how the positions of two bicycles (called A and B) change as time passes. The units of position are miles; the units of time are hours.



(a) Which bike is moving faster at $t = \frac{1}{4}$ (that is, after 15 minutes)? How do you know?

Solution: Bike A; its position is changing more quickly.

(b) Which bike is moving faster at the end of the ride (at $t = 1$)?

Solution: Bike B.

(c) Do the bikes finish the hourlong ride together, or does one bicyclist beat the other? How can you tell?

Solution: Together; they are both 10 miles away from the start at $t = 1$.

Notice that a steeper curve on the graph corresponds to a higher velocity. A steep curve means that the position is changing quickly, which means the bike is moving fast. Refer to the graph on Page 1 to answer the following questions.

2. (a) According to the graph, during **the last half-hour of the bike ride**, when is bike A moving the fastest?

   Solution: When the slope of its graph is the steepest between $t = \frac{1}{2}$ and $t = 1$. That is, at $t = \frac{1}{2}$.

   (b) At about what time does bike B start catching up with bike A? That is, when does the distance between bikes A and B start to shrink?

   Solution: Somewhere around $t = \frac{3}{8}$.

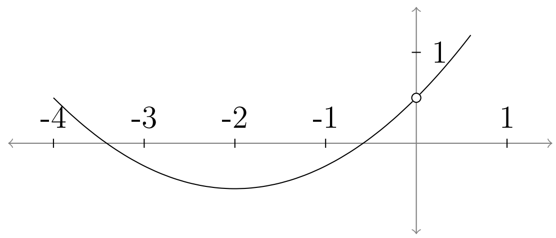
   (c) Do you think there is an a time when bikes A and B are moving at exactly the same velocity? Either estimate that time by looking at the graph, or explain why there can't be such a time.

   Solution: Yes, there would be such a time. Rephrase the question: is there a time t so that the tangent line to the curve for A is parallel to the tangent line to the curve for B? Try to see geometrically why the answer is yes.

   (d) Are the questions and answers to parts (2b) and (2c) related? Why or why not?

   Solution: Bike B is catching up with bike A means that B is travelling faster than A. Since B started out travelling slower than A, Its velocity went from less than the velocity of A to greater than the velocity of A. The instant it made the transition was the time the velocities were equal.

3. Now consider the graph of $f(x)$ below:

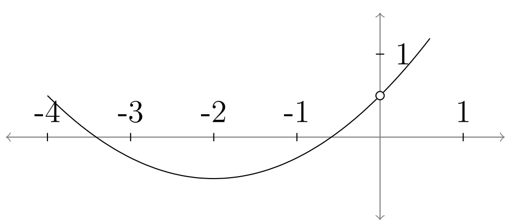


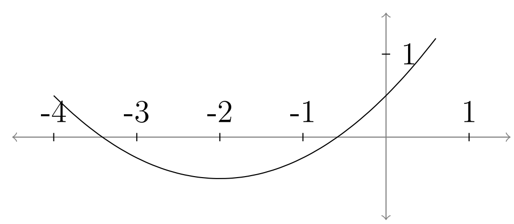
The function $f(x)$ isn't defined at $x = 0$, but you can still find $\lim_{x \to 0} f(x)$ by looking at the graph. What is this limit?

Solution: $\lim_{x \to 0} f(x) = \frac{1}{2}$.

It would be nice not to need to look at a graph to find limits like these, since many functions are very difficult to graph. Fortunately, there is a method that often works for computing such limits. This method, described below, will come up a *lot* in Math 124.

4. Below there are two equations and two graphs. Which equation corresponds to which graph?





$$y = \frac{\frac{1}{4}x^3 + x^2 + \frac{1}{2}x}{x} \qquad y = \frac{1}{4}x^2 + x + \frac{1}{2}$$

Draw lines connecting each equation to its graph. How do you know your answer is correct?

Solution: The left equation corresponds to the left graph, since there can be no point with x-coordinate 0.

Now, let $f(x) = (\frac{1}{4}x^3 + x^2 + \frac{1}{2}x)/x$, and let $g(x) = \frac{1}{4}x^2 + x + \frac{1}{2}$. The graphs of $f(x)$ and $g(x)$ above are identical except at $x = 0$: there $g(x)$ is defined and $f(x)$ is not.

5. Why are the graphs of $f(x)$ and $g(x)$ identical except at $x = 0$? (If you have trouble seeing the reason, try plugging in specific non-zero x's. What goes wrong when you try to plug in $x = 0$?)

> **Solution:** If x is not 0, you can just factor out an x from the numerator and cancel it with the x in the denominator, obtaining the expression for $g(x)$.

6. You calculated $\lim\limits_{x\to 0} f(x)$ above in Question 3 by looking at its graph. How does this limit compare to the value $g(0)$?

> **Solution:** $\lim\limits_{x\to 0} f(x) = \frac{1}{2} = g(0)$.

Let's review: we had a function $f(x)$ that had a hole at $x = 0$, and we hoped to find $\lim\limits_{x\to 0} f(x)$ without needing to refer to a graph. We did this by using a function $g(x)$ that is exactly the same as $f(x)$ except that it *is* defined at $x = 0$. That is, we filled in the gap in the graph of $f(x)$. Then we could just plug 0 into $g(x)$ to find the limit. This method works because the new function g is *continuous*. You will see continuous functions later in class.

7. Find $\lim_{x\to -1} \frac{x^2-x-2}{x+1}$. Where is the function not defined? How can you fix it?

> **Solution:** First, factor the numerator:
>
> $$\frac{(x+1)(x-2)}{x+1}.$$
>
> Now cancel the factors of $(x + 1)$, obtaining a new function that matches the old one everywhere except that it *is* defined at $x = -1$: $x - 2$. Now we can just plug in $x = -1$: we get that the limit is -3.