Chapter 4

Part 1

Linear modelling

Lines and circles

Word problems
Linear modelling

Clue words: linear, constant rate

Goal: find equation of a line $y = mx + b$ (if not vertical) and use it to answer questions in the problem
Yearly resident tuition at the UW is tabulated below:

<table>
<thead>
<tr>
<th>Year</th>
<th>Tuition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1989</td>
<td>$1827</td>
</tr>
<tr>
<td>1995</td>
<td>$2907</td>
</tr>
</tbody>
</table>

- Assume tuition at UW increases at a constant rate. When will tuition be $20,000 a year?

Set variables \( x, y \)

\[ y = \text{tuition} \times \text{year} \quad (\text{after 1989}) \]
What kind of line problem is it?

We are given two points P(0, 1827)
Q(6, 2907)

Equation of line PQ is

\[
y - 1827 = \frac{2907 - 1827}{6} x
\]

or \[y = 1827 + 180x\]

Solve \[20000 = 1827 + 180x\]
\[ \frac{20000 - 1827}{180} = x \]

\[ x \approx 100.96 \approx 101 \]

in the year 2090

* If tuition at Ux is predicted by the formula:

\[ \text{tuition} = 1000 + 300x, \]

where \( x \) is measured in years after 1989 when will tuition at
UX be $1000 more expensive than at UW?

\[ y_{uw} = 180x + 1827 \]

\[ y_{ux} = 300x + 1000 \]

(compare slopes and y intercepts)

We want \( y_{ux} = 1000 + y_{uw} \)

So

\[ 300x + 1000 = 1000 + 180x + 1827 \]

\[ x = \frac{1827}{300 - 180} \approx 15.24 \text{ or in } 2004 \]
Tuition at the UW increases at a constant rate of $500 per year. Tuition was $4000 a year in 2000, when will tuition exceed $15000 a year?

\[ y = \text{tuition} \]
\[ t = \text{years after 2000} \]
\[ y = 500t + 4000 \]

\[ 15000 = 500t + 4000 \]
\[ t = \frac{11000}{500} = 22 \]

After 2022
A crop dusting airplane flying a constant speed of 120mph is first spotted 2 miles South and 1.5 miles East of the center of circular irrigated field. The irrigated field has radius 1 mile. The plane flies in a straight line to a point 1 mile West of the center of the irrigated field.

1. Find the location \( P \) where the crop duster enters the airspace above the field

2. When does the plane first enters the airspace above the field? (Assume time \( t=0 \) corresponds to when the plane is first spotted)

3. How much time does the plane spend flying over the irrigated field?

4. How close does the plane get to the center of the field?

5. Where should the plane enter the field if we want it to get no closer than 0.5 miles to the center (still flying in a straight path)?
1) Find equation of line PQ (path of the plane).

Q (-1, 0)  P (1.5, -2)

\[
\frac{y + 2}{x - 1.5} = \frac{2}{-1 - 1.5}
\]

\[
y + 2 = -\frac{4}{5} (x - 1.5)
\]
2) Find intersections of path PQ and circle (where plane enters and leaves the airspace above the field)

\[ \begin{align*}
    x^2 + y^2 &= 1 \\
    y + 2 &= -\frac{4}{5}(x - 1.5) \\
    \end{align*} \]

\[ \begin{align*}
    x^2 + \left(-2 - \frac{4}{5}(x - 1.5)\right)^2 &= 1 \\
    y &= -2 - \frac{4}{5}(x - 1.5) \\
    \end{align*} \]
\[ y = -2 - \frac{4}{5} (x - 1.5) \]  
\[ 1.64x^2 + 1.28x - 0.36 = 0 \]

\[ x = -1, \quad 0.22 \]

(Quadratic formula):

\[ ax^2 + bx + c = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ y = 0, \quad -0.97 \]

Enters at \( A(0.11, -0.97) \)

Exits at \( Q(-1, 0) \)
3) Find distance PA
P (1.5, -2) A (0.22, -0.97)

\[ \sqrt{(0.22 - 1.5)^2 + (-0.97 + 2)^2} \]

= 1.64 mi

4) Find time to cover distance PA

\[ t = \frac{d}{v} = \frac{1.64}{120} = \]

= 0.01 hours

or \[ \frac{1}{100} = \frac{36}{3600} \text{ sec} \]
5) Find distance $AQ$

$$\sqrt{(0.22 + 1)^2 + (-0.97)^2} = $$

1.56 mi

6) Find time $t$

necessary to cover

distance $AQ$

$$t = \frac{d}{v} = \frac{1.56}{120} \approx 0.01$$

7) We have to find

the distance between
0 (0, 0) center of field
and line \( y + 2 = -\frac{4}{5} (x - 1.5) \)
plane path

Method 1

Distance of \((x_1, y_1)\) from \(ax + by + c = 0\)
is \( d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \)
so rewrite eq of line as

\[
\frac{4}{5} x + y + 0.2 = 0 \quad \text{c}(0, 0)
\]

\[
d = \frac{|4 \cdot 0 + 0 + 0.2|}{\sqrt{(\frac{4}{5})^2 + 1}} \approx 0.62
\]
Method 2

\[
\text{or } 0.62 \times 5280 = 3273.6 \text{ feet}
\]

a) Find equation of line \( l_2 \) through \( P \)

b) Find intersection of \( l_1 \) and \( l_2 \)

c) Calculate \( d = d(P, a) \)

In our problem

\[
P(0, 0)
\]

\[
l_1: y = -2 - \frac{4}{5}(x - 1.5)
\]
\[ E_2 = y = \frac{5}{4} x \]

Intersection

\[
\begin{align*}
  y &= \frac{5}{4} x \\
  y &= 2 - \frac{4}{5} (x - 1.5)
\end{align*}
\]

has solution

\[ x = -0.3902 \quad y = -0.4878 \]

\[ d = \sqrt{(-0.3902)^2 + (-0.4878)^2} \]

\[ d = (0.62 \text{ mi}) \]

\[ Q (-0.3902, -0.4878) \]
Last question can be solved by finding the lines tangent to the circle centered at (0,0) and with radius 0.5 through P. There are two such lines $l_1$ and $l_2$. The intersections $Q_1$ and $Q_2$ of $l_1$ and $x^2 + y^2 = 1$ and
$l_2$ and $x^2 + y^2 = 1$

are the points

where the plane

should enter the

airspace.
\[ p \left( 1.5, -2 \right) \]

\[ x^2 + y^2 = 1.5 \]

We want to find the equations of tangent lines to \( C \) through \( p \).

First find \( R_1 \) and \( R_2 \).

Slope of \( OR_2 = \frac{y}{x} \)

\[ m_1 = -\frac{x}{y} \]

\[ m_2 = \frac{y+2}{x-1.5} \]
\[
\begin{align*}
\begin{cases}
x^2 + y^2 &= \frac{1}{4} \\
-x &= \frac{y + 2}{y} \\
x - 1.5x &= y + 2y
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
x^2 + y^2 &= \frac{1}{4} \\
1.5x - 2y &= x^2 + y^2
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
x^2 + y^2 &= \frac{1}{4} \\
1.5x - 2y &= \frac{1}{4}
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
x^2 + y^2 &= \frac{1}{4} \\
1.5x - 1.125 &= y \\
\frac{9}{8} &= 3/4
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
x^2 + (\frac{3}{4}x - \frac{1}{8})^2 &= \frac{1}{4} \\
\frac{3}{4}x - \frac{1}{8} &= y
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
3y &= 4x - 118 \\
25x^2 - 3x - 15 &= 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
x &= 0.45 \\
y &= 0.21
\end{cases}
\quad
\begin{cases}
x &= -0.33 \\
y &= -0.37
\end{cases}
\end{align*}
\]

\[
C: x^2 + y^2 = 1
\]
To find $Q_1$

write equation of line $l$

\[ \frac{y - 0.21}{-2 - 0.21} = \frac{x - 0.45}{1} \]

\[ l: y = 0.21 - 2.1(x - 0.45) \]

Complete intersections of $l$ and $C$

\[
\begin{align*}
\begin{cases}
y = 0.21 - 2.1(x - 0.45) \\
x^2 + y^2 = 1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
y = 0.21 - 2.1(x - 0.45) \\
x^2 + (0.21 - 2.1(x - 0.45))^2 = 1
\end{cases}
\end{align*}
\]
\[ \begin{align*}
2y &= 0.21 - 2.1(x - 0.45) \\
5.41x^2 - 4.85x + 0.33 &= 0
\end{align*} \]

\[ \begin{align*}
x &= 0.82, 0.07 \\
y &= -0.567, 1.008
\end{align*} \]

What solution do I want? Look at picture

\[ Q_1 = (0.82, -0.567) \]

Plane can enter at \( Q_1 \).

We can also do similar calculations to find \( Q_2 \) (I will skip it) and plane could enter at \( Q_2 \).