Ch 11

Exponential modelling

The number e

(Compound interest)

A bacteria population grows exponentially. At time $t=1$ there are 2000 bacteria. At time $t=2$ there are 3000 bacteria. How many bacteria are there at time $t=3$?

Find exponential function through $(1, 2000)$ and $(2, 3000)$

$$f(t) = A_0 a^t$$

Set up a system

$$\begin{cases} 2000 = A_0 a^1 \\ 3000 = A_0 a^2 \end{cases}$$

$$\begin{cases} \frac{2000}{a} = A_0 \\ 3000 = \frac{2000}{a} a^2 = 2000a \end{cases}$$

$$\begin{cases} a = \frac{3}{2} \\ A_0 = \frac{2000}{\frac{3}{2}} = \frac{4000}{3} \end{cases}$$

$$f(t) = \frac{4000}{3} \left(\frac{3}{2}\right)^t$$

$$f(3) = \frac{4000}{3} \left(\frac{3}{2}\right)^3 = 4500$$

A bacteria population grows exponentially

Intially there are 5000 bacteria. It takes 1.3 hours to double.

How many bacteria will there be after 2 hours?

Doubling time 1.3

through (0, 5000)

$$f(t) = A_0 \, a^t$$

$$5000 = A_0$$

$$1.3 = \frac{\ln 2}{\ln a}$$

$$\ln a = \frac{\ln 2}{1.3}$$

$$e^{\ln a} = e^{\frac{\ln 2}{1.3}}$$

$$a \approx 1.7044$$

$$f(t) = 5000 \cdot (1.7044)^t$$

$$f(2) = 5000 \cdot (1.7044)^2 \approx 14525$$

Note $A_0 = f(0)$

A bacteria population grows exponentially. If it takes 2 hours to double, how long will it take to increase 5 times?

$f(t) = A_0 a^t$

$2 = \frac{\ln 2}{\ln a}$

$\ln a = \frac{\ln 2}{2}$, $a = e^{\frac{\ln 2}{2}} =$

$= 2^{1/2} = \sqrt{2}$

so $f(t) = A_0 \sqrt{2}^{\,t}$

Note: A_0 is not known

Now we went population to increase 5 times

$$5A_0 = A_0 (\sqrt{2})^t$$

$$5 = (\sqrt{2})^t$$

$$\frac{\ln 5}{\ln \sqrt{2}} = t$$

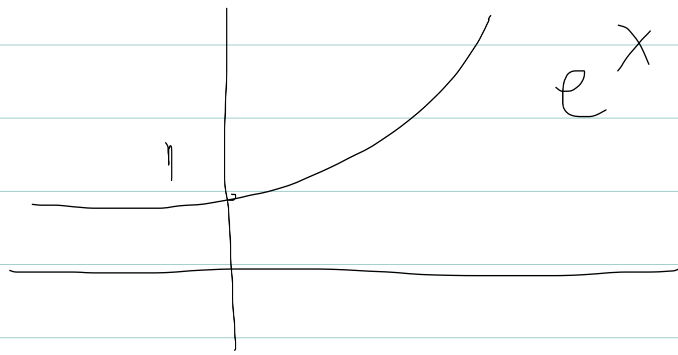
Formula: given

$f(t) = A_0 a^t$

time it takes to increase n times is

$$t = \frac{\ln n}{\ln a}$$

$e = 2.71828 \ldots\ldots > 1$



$a = e^{\ln a}$

So we can write

$f(x) = A_0 a^x$ as

$A_0 e^{\ln(a) x}$ or

$f(x) = A_0 e^{kx}$

Convert

$f(x) = \frac{2}{3^{x-4}}$ to

"e form"

First put f int standard form

$f(x) = 2 \frac{1}{3^x \cdot 3^{-4}} =$

$= 2 \cdot 81 \cdot \left(\frac{1}{3}\right)^x$

$= 162 \cdot e^{(\ln \frac{1}{3}) \cdot x}$

Nichifor Spring 2012

$\boxed{4}$ (13 pts) In 1990, the U.S. minimum wage was \$3.80 per hour. In 1997, it was \$5.15 per hour. Assume the minimum wage grows according to an exponential model $W(t)$, where t represents the number of years after 1990.

a) (6 pts) Find a formula for $W(t)$.

$P(0, 3.8)$ $Q(7, 5.15)$

$f(t) = A_0 a^t$

$\begin{cases} 3.8 = A_0 \\ 5.15 = A_0 a^7 \end{cases}$ $\begin{cases} 3.8 = A_0 \\ 5.15 = 3.8 a^7 \end{cases}$ $a = \sqrt[7]{\frac{515}{380}}$

b) (2 pts) What does the model predict for the current minimum wage? (year 2012)

$f(t) = 3.8\left(\sqrt[7]{\frac{515}{380}}\right)^t$

$f(22) \approx 9.88$

c) (5 pts) In what year is the minimum wage expected to reach \$100 per hour, according to this model?

$100 = 3.8 \sqrt[7]{\frac{515}{380}}^{\,t}$ so ℓ for t

$\frac{100}{3.8} = \sqrt[7]{\frac{515}{380}}^{\,t}$

$\ln\left(\frac{1000}{38}\right) = \ln\left(\sqrt[7]{\frac{515}{380}}^{\,t}\right)$

$t \approx \ln(1000/38) / \ln\left(\sqrt[7]{\frac{515}{380}}\right)$

$t \approx 75.3$ in 2065

Interest

If a sum P_0 is invested at a (yearly) interest rate r compounded yearly, then the value of the investment after t years is

$$P(t) = \boxed{P_0(1+r)^t}$$

NOTE: r is a decimal

so if interest is 2% $r = 0.02$

If the interest is compounded n times a year after t years we have

$$P(t) = \boxed{P_0\left(1+\frac{r}{n}\right)^{nt}}$$

If the interest is compounded continuously after t years we have

$$\boxed{P(t) = P_0\, e^{rt}}$$

Final Sp 2012

Problem 3 (12 points)

The population of Arcadia increases by 8% every 10 years. The population of Brom triples every 120 years.

The two cities had equal populations of 10,000 residents each in the year 2000.

In what year will the city of Brom have twice as many residents as the city of Arcadia?

Let $t=0$ in 2000

Find formula for $A(t)$ population of Arcadia.

Method I SKIP

$$A(t) = 10{,}000(1+0.08)^{t/10}$$

(like interest compounded every 10 years so $n=\frac{1}{10}$)

Method 2 Read

$(0, 10000)$ $(10, 10800)$

$$A(t) = A_0 a^t$$

$$\begin{cases} 10000 = A_0 \\ 10800 = A_0 a^{10} \end{cases} \qquad 10000 = A_0, \quad \sqrt[10]{\frac{10800}{10000}} = a$$

$$A(t) = 10\,000\left(\sqrt[10]{1.08}\right)^t$$

Find a formula for $B(t)$ (the population of Brom); $B(t) = B_0 b^t$

$A_0 = 10,000$

$$120 = \frac{\ln 3}{\ln a}, \quad \ln a = \frac{\ln 3}{120}$$

$$a = e^{\frac{\ln 3}{120}} = e^{\ln 3 \cdot \frac{1}{120}} =$$

$$= 3^{1/120} = \sqrt[120]{3}$$

$$B(t) = 10\,000\left(\sqrt[120]{3}\right)^t$$

Note $B(t)$ goes through $(0, 10,000)$ $(120, 30,000)$

We need $B(t) = 2A(t)$

$$10000\sqrt[120]{3}^{\,t} = 2\cdot 10000\sqrt[10]{1.08}^{\,t}$$

$$\frac{\sqrt[120]{3}^{\,t}}{\sqrt[10]{1.08}^{\,t}} = 2$$

$$\left(\frac{\sqrt[120]{3}}{\sqrt[10]{1.08}}\right)^{t} = 2$$

$$t = \frac{\ln 2}{\ln\left(\frac{\sqrt[120]{3}}{\sqrt[10]{1.08}}\right)} \approx 475$$

in 2475