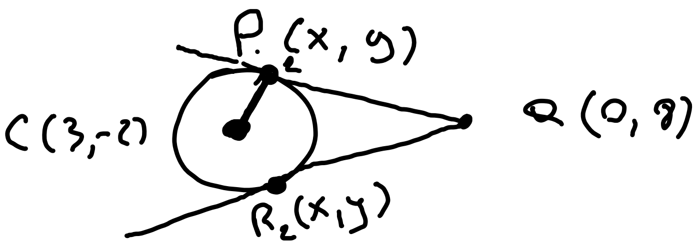
Lesson 4

Read Chapter 4

Linear modeling

Lines and Circles word problems

Find the tangent to to the circle $(x-3)^2 + (y+2)^2 = 5$ through the point $Q(0,8)$



Want to find P point of tangency

$$\begin{cases} \text{Circle} \\ \text{write } m \text{ slope of tangent in two different ways} \end{cases}$$

$$\begin{cases} (x-3)^2 + (y+2)^2 = 5 \\ (m =) \dfrac{y-8}{x-0} = -\dfrac{1}{\frac{y+2}{x-3}} = -\dfrac{x-3}{y+2} = \dfrac{3-x}{y+2} \end{cases}$$

slope of line CP $\quad \dfrac{y+2}{x-3}$

$$\begin{cases} (x-3)^2 + (y+2)^2 = 5 \\ \dfrac{3-x}{2+y} = \dfrac{8-y}{-x} \end{cases}$$

Expand squares

cross multiply

$$\begin{cases} x^2 - 6x + 9 + y^2 + 4y + 4 = 5 \\ -3x + x^2 = 16 + 8y - 2y - y^2 \end{cases}$$

simplify

simplify

$$\begin{cases} x^2 + y^2 = 6x - 4y - 8 \\ x^2 + y^2 = 16 + 6y + 3x \end{cases}$$

Keep

replace

$$\begin{cases} x^2 + y^2 = 6x - 4y - 8 \\ 6x - 4y - 8 = 16 + 6y + 3x \end{cases}$$

solve for y

(or x)

$$\begin{cases} x^2 + y^2 = 6x - 4y - 8 \\ \dfrac{3x - 24}{10} = \dfrac{\cancel{10}}{\cancel{10}} y \end{cases}$$

replace y

keep

simplify

$$\begin{cases} x^2 + \left(\dfrac{3x - 24}{10}\right)^2 = 6x - 4\left(\dfrac{3x - 24}{10}\right) - 8 \\ y = \dfrac{3x - 24}{10} \end{cases}$$

keep

$$\begin{cases} x^2 + \dfrac{9x^2 - 2 \cdot 3 \cdot 24x + 24^2}{100} = 6x - \dfrac{12x - 96}{10} - 8 \\ \vdash \mid \end{cases}$$

$$\underbrace{109}_{a} x^2 \underbrace{- 624}_{b} x \underbrace{+ 416}_{c} = 0$$

$x \approx 4.95$

$x = 0.77$

$$y = \frac{3x - 24}{10}$$

For $x = 4.9545$ $\quad y \approx -0.91$ $\quad P_1$

For $x = 0.7703$ $\quad y \approx -2.17$ $\quad P_2$

quadratic formule: solutions of

$ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2a}$$

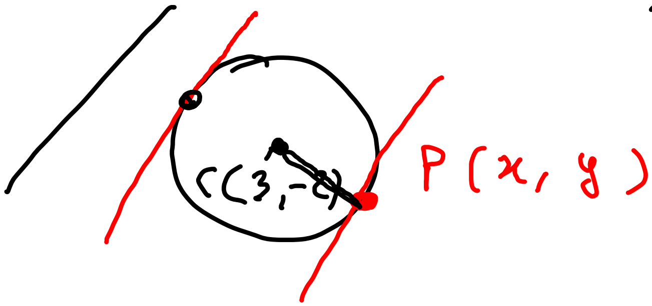
tangent 1 line through $Q(0,8)$ $P_1(4.95, -0.91)$

$$y = 8 + \frac{8+0.91}{-4.95}x \qquad \boxed{y = 8 - 1.8x}$$

tangent 2 : line through $Q(0,8)$ $P_2(0.77, -2.17)$

$$y = 8 + \frac{8+2.17}{-0.77}x \qquad \boxed{y = 8 - 13.21x}$$

Find the equation of the line tangent to to the circle $(x-3)^2 + (y+2)^2 = 5$ and parallel to the line $4x - 2y + 10 = 0$ $\rightarrow$ $\frac{+2y}{2} = \frac{+4x+10}{2}$ $\boxed{y = 2x + 5}$



$$\begin{cases} (x-3)^2 + (y+2)^2 = 5 \\ -\dfrac{1}{\frac{y+2}{x-3}} = 2 \end{cases}$$

slope CP : $\frac{y-(-2)}{x-3} = \frac{y+2}{x-3}$

$$\begin{cases} (x-3)^2 + (y+2)^2 = 5 \\ \dfrac{3-x}{y+2} = 2 \end{cases}$$

$$\begin{cases} (x-3)^2 + (y+2)^2 = 5 \\ 3 - x = 2y + 4 \end{cases} \qquad \begin{cases} (x-3)^2 + (y+2)^2 = 5 \\ x = -2y - 1 \end{cases}$$

$$(\underbrace{-2y-1}_{x} - 3)^2 + (y+2)^2 = 5$$ This is a quadratic in y. simplify and solve

$y^2 + 4y + 3 = 0$ has sol $y = -3$, $y = -1$

so points of tangency are: $x = -2(-3) - 1 = 5$, $y = -3$ $(5, -3)$

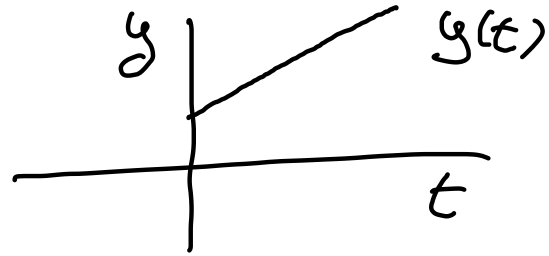
$x = -2(-1) - 1 = 1$ $y = -1$ $(1, -1)$ Tangent lines are

① $y = -1 + 2(x-1)$ ② $y = -3 + 2(x-5)$

Linear modelling

Clue words: LINEAR , CONSTANT RATE
Goal: find the equation of a line and use it to answer questions in the problem

Yearly resident tuition at the UW was \$1827 in 1989 and \$2907 in 1995. Assume tuition increases at a constant rate. When will tuition be \$20000 a year ?



$y(t)$: UW tuition at year t

$$y(t) = mt + b$$

$P(\underset{t}{1989}, \underset{y}{1827})$ $Q(1995, 2907)$

$t = 0$ corresponds to 1989

$A(0, 1827)$ $B(6, 2907)$

$$y = 1827 + \frac{2907 - 1875}{6 - 0} \cdot t$$

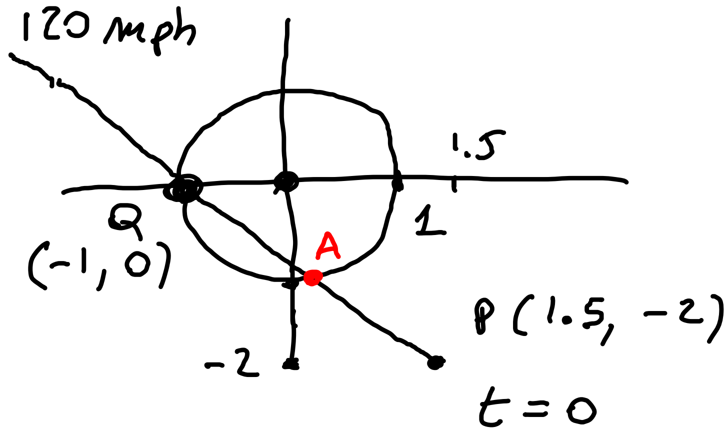
$$y = 1827 + 180t$$

$$20{,}000 = 1827 + 180t \qquad t = \frac{20000 - 1827}{180} \approx 101$$

$\boxed{2090}$

A crop dusting airplane flying a constant speed of 120mph is first spotted 2 miles South and 1.5 miles East of the center of cicular irrigated field. The irrigated field has radius 1 mile. The plane flies in a straight line to a point 1 mile West of the center of the irrigated field.

Find the location A where the crop duster enters the airspace above the field.



① Find line PQ

② Find intersection of line PQ and $x^2+y^2=1$

$$y = -2 + \frac{-2-0}{1.5-(-1)}(x-1.5)$$

$$y = -2 - 0.8\cdot(x-1.5)$$

$$y = -0.8x - 0.8$$

$$\begin{cases} x^2 + y^2 = 1 \\ y + 2 = -0.8(x - 1.5) \end{cases} \qquad \begin{cases} y = -0.8x - 0.8 \\ x^2 + (-0.8x - 0.8)^2 = 1 \end{cases}$$

$$x^2 + 0.64x^2 + 0.64 - 1.28x - 1 = 0$$

$$1.64x^2 + 1.28x - 0.36 = 0$$

$x = -1$, $\qquad$, $x = 0.22$

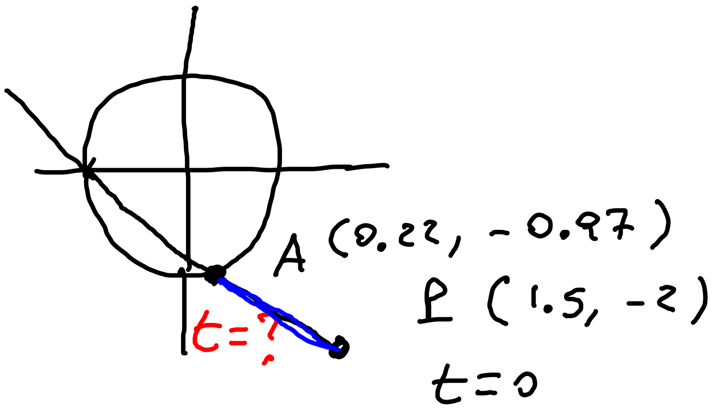
$y = -0.8(-1) - 0.8 = 0$ $\qquad$ $y = -0.8 \cdot 0.22 - 0.8 = -0.97$

Enters at $A\ (0.22, -0.97)$

Exits at $Q\ (-1, 0)$

When does the plane first enter the airspace above the field ?
(Assume time t=0 corresponds to when the plane is first spotted)

$V = 120$ mph



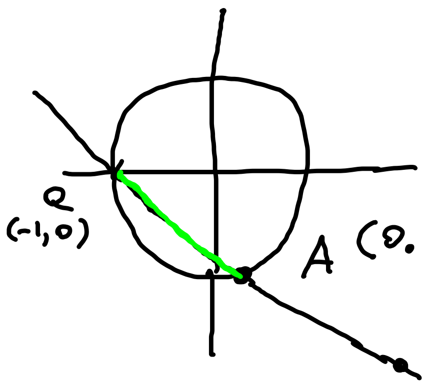
$$d = v \cdot t$$

$$\frac{d}{v} = t$$

$$d(P, A) = \sqrt{(1.5 - 0.22)^2 + (-2-(-0.97))^2} = 1.64$$

$$t = \frac{1.64}{120} = 0.014 \text{ h.}$$

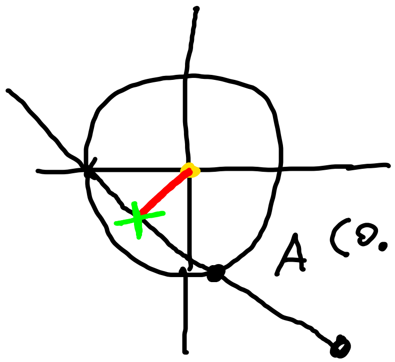
How much time does the plane spend flying over the irrigated field?



$$t = \frac{d}{v} = \frac{d(AQ)}{120} = \frac{\sqrt{(0.22+1)^2 + (-0.97)^2}}{120}$$

$$\approx 0.013$$

How close does the plane get to the center of the field?



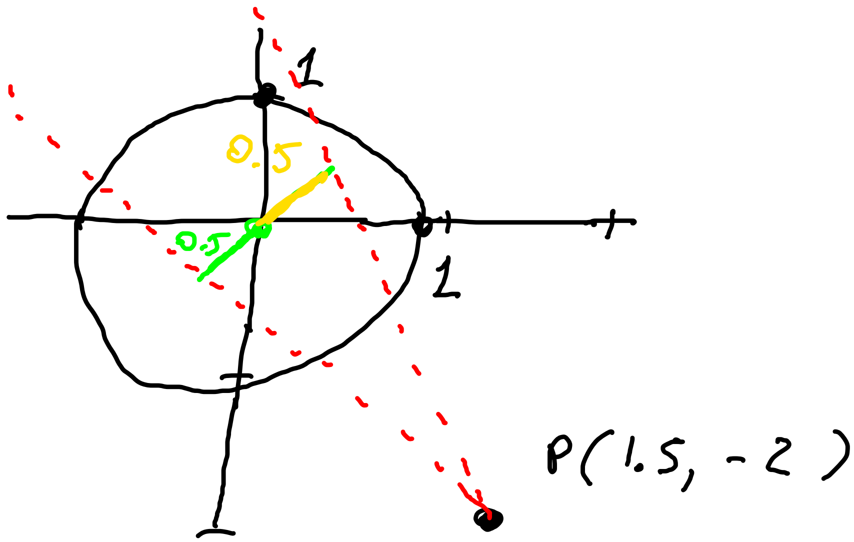
$A\,(0.22, -0.97)$

$P\,(1.5, -2)$

$y = -0.8x - 0.8 \rightarrow 0.8x + y + 0.8 = 0 \qquad C(0,0)$

$$d = \frac{|0.8\cdot 0 + 0 + 0.8|}{\sqrt{0.8^2 + 1^2}} = 0.62 \text{ mi}$$

Where should the plane enter the field if we want it to get 0.5 miles to the center (still flying in a straight path), but no closer?



Next time

$$\begin{cases} x^2+y^2=\frac{1}{4} \\ -\frac{x}{y}=\frac{y+2}{x-1.5} \end{cases} \qquad \begin{cases} x^2+y^2=1/4 \\ -x^2+1.5x=y^2+2y \end{cases}$$

$$\begin{cases} x^2+y^2=1/4 \\ 1.5x-2y=x^2+y^2 \end{cases}$$

$$\begin{cases} x^2+y^2=1/4 \\ 1.5x-2y=\frac{1}{4} \end{cases}$$

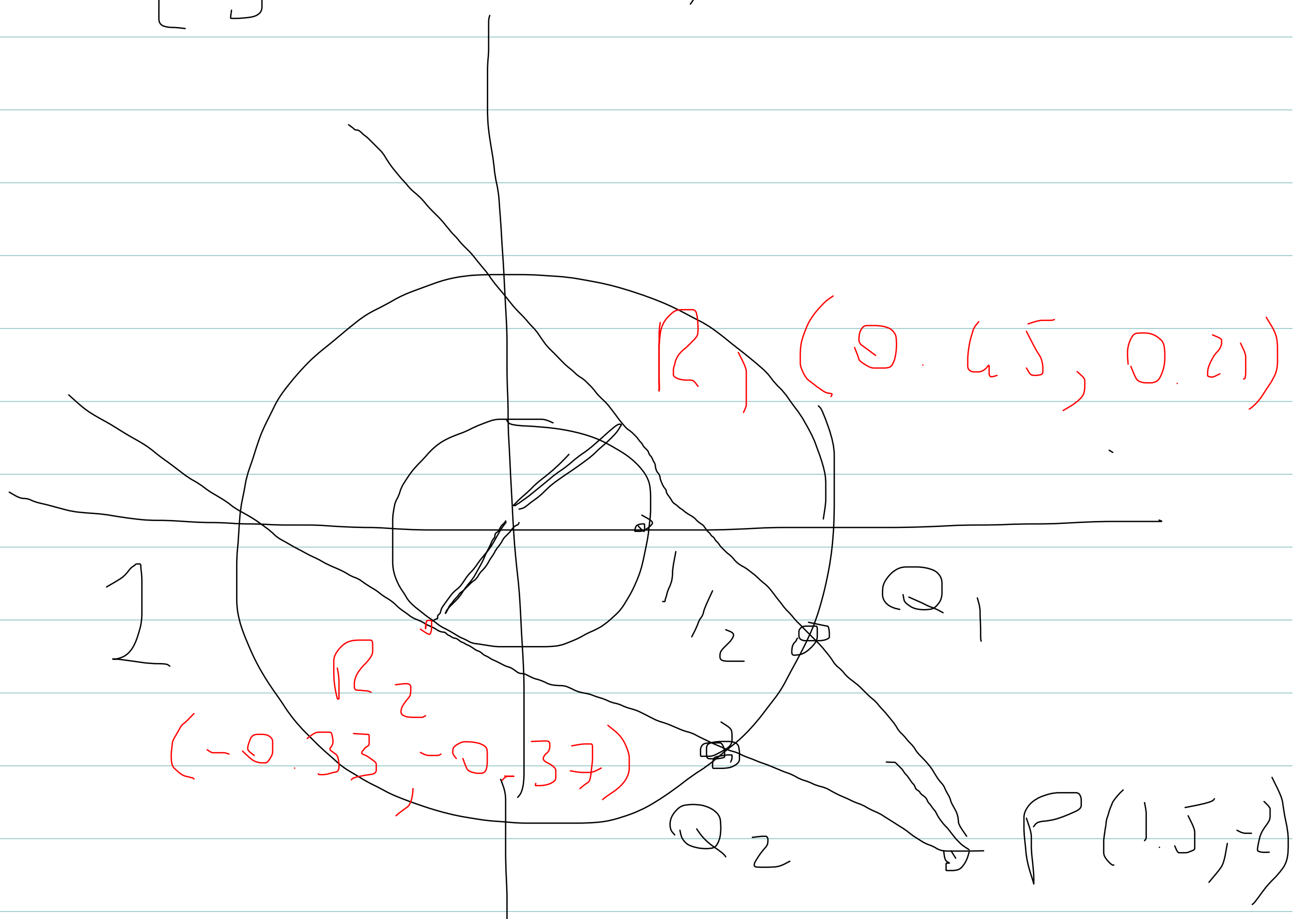
$$\begin{cases} x^2+y^2=1/4 \\ \frac{1.5}{2}x-1/8=y \end{cases}$$

$\frac{1.5}{2} = 3/4$

$$\begin{cases} x^2+\left(3/4\,x-\frac{1}{8}\right)^2=1/4 \\ 3/4\,x-\frac{1}{8}=y \end{cases}$$

$$\begin{cases} y = \frac{3}{4}x - \frac{1}{8} \\ 25x^2 - 3x - 15 = 0 \end{cases}$$

$$\begin{cases} x = 0.45, \ -0.33 \\ y = 0.21, \ -0.37 \end{cases}$$



$C: x^2 + y^2 = 1$

First tangent line y_1 is line ℓ_1 through $P(1.5, -2)$ and $R_1(0.45, 0.21)$

$$y = 0.21 + \frac{-2-0.21}{1.5-0.45}(x-0.45)$$

$$y = 0.21 - 2.1(x-0.45)$$

Q_1 is intersection of ℓ_1 and circle $x^2+y^2=1$

$$\begin{cases} y = 0.21 - 2.1(x-0.45) \\ x^2+y^2=1 \end{cases}$$

$$\begin{cases} y = 0.21 - 2.1(x-0.45) \\ x^2 + (0.21 - 2.1(x-0.45))^2 = 1 \end{cases}$$

$$\begin{cases} y = 0.21 - 2.1\,(x - 0.45)^2 \\ 5.41x^2 - 4.85x + 0.33 = 0 \end{cases}$$

$$\begin{cases} x = 0.82\ ,\ 0.07 \\ y = -0.567\ ,\ 1.008 \end{cases}$$

What solution do I want? Look at picture

$Q_1 = (0.82, -0.567)$

Plane can enter at Q_1

We can also do similar calculations to find Q_2 (I will skip it) and plane could enter at Q_2