Lesson 3

Read Chapter 3

Circles

Find the point on the line $y = 2x + 1$ that is closest to the point $P(3, 0)$

① Find line through $(3,0)$ $\perp$ to $y = 2x+1$

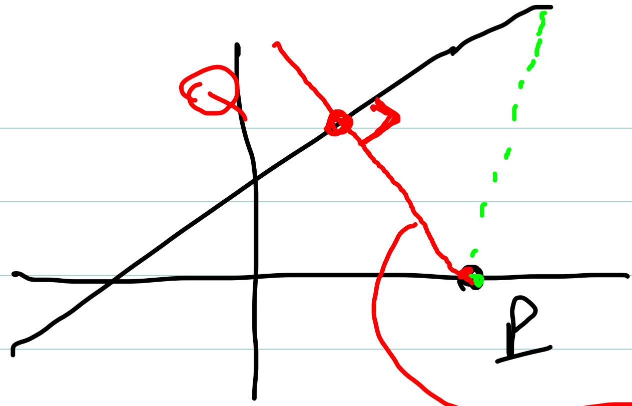
$y = -\frac{1}{2}(x-3)$

② Find intersection of $y = 2x+1$ and $y = -\frac{1}{2}(x-3)$

$Q\left(\frac{1}{5}, \frac{7}{5}\right)$ (Calculations next page)

③ Find $d(P,Q) = \sqrt{\left(3-\frac{1}{5}\right)^2 + \left(0-\frac{7}{5}\right)^2} = \sqrt{\frac{14^2}{5^2} + \frac{7^2}{5^2}} = \sqrt{\frac{245}{5^2}} = \frac{\sqrt{245}}{5} = \frac{\sqrt{7^2 \cdot 5}}{5}$

$= \frac{7}{5}\sqrt{5}$



→ shortest distance

Calculations for (2)

$$y = 2x+1$$
$$y = -\frac{1}{2}(x-3)$$

$$2x+1 = -\frac{1}{2}(x-3)$$
$$2x + \frac{1}{2}x = \frac{3}{2} - 1$$
$$\frac{5}{2}x = \frac{1}{2}$$
$$\boxed{x = \frac{1}{5}}$$

$$y = 2 \cdot \frac{1}{5} + 1 = \boxed{\frac{7}{5}}$$

Other formula:

Distance between $P(x_0, y_0)$ and $Ax+By+C=0$ is

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}$$

For $y = 2x+1$ $\quad P(3, 0)$

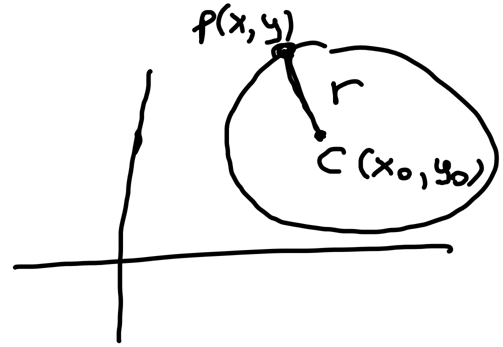
$$0 = \underbrace{2}_{A}x \underbrace{-}_{B} y + \underbrace{1}_{C}$$

$$d = \frac{|2\cdot 3 - 0 + 1|}{\sqrt{2^2+(-1)^2}} = \frac{|7|}{\sqrt{5}} = \frac{7}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{7}{5}\sqrt{5}$$

Useful facts about circles

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = r$$



1. Equation of a circle (in standard form):

   $$(x-x_0)^2+(y-y_0)^2=r^2$$

   The circle has center C (x_0, y_0) and radius r.
2. If a line L is tangent to a circle at P, then the line is perpendicular to the radius CP.

Given a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ and a point $P(x_1, y_1)$

P is on the circle if $(x_1 - x_0)^2 + (y_1 - y_0)^2 = r^2$

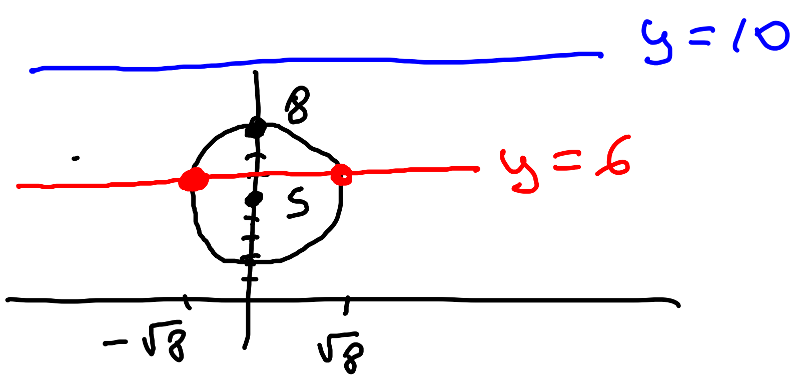
P is inside the circle if $(x_1 - x_0)^2 + (y_1 - y_0)^2 < r^2$

P is outside the circle if $(x_1 - x_0)^2 + (y_1 - y_0)^2 > r^2$

For which value of x is $P(x, 6)$ on the circle centered at (0,5) with radius 3?

$(x-0)^2 + (y-5)^2 = 3^2$

$x^2 + (6-5)^2 = 9$ solve for x : $x^2 = 9-1 = 8$; $x = \pm\sqrt{8}$



For which value of x is $P(x, 10)$ on the circle centered at (0,5) with radius 3?

Find the center and radius of the circle

$$(x-x_0)^2+(y-y_0)^2=r^2$$

$$x^2+6x+y^2-2y+9=0$$

In general if you start with $x^2+ax+y^2+by+c=0$ write

$$\left(x+\frac{a}{2}\right)^2+\left(y+\frac{b}{2}\right)^2+c-\frac{a^2}{4}-\frac{b^2}{4}=0$$

$$x^2+\cancel{2}\cdot x\cdot\frac{a}{\cancel{2}}+\cancel{\frac{a^2}{4}}+y^2+\cancel{2}y\frac{b}{\cancel{2}}+\cancel{\frac{b^2}{4}}+c-\cancel{\frac{a^2}{4}}-\cancel{\frac{b^2}{4}}=0$$

Note $(A+B)^2=A^2+2\cdot A\cdot B+B^2$

In our case: $(x+3)^2+(y-1)^2+\underbrace{9-\frac{6^2}{4}-\frac{(-2)^2}{4}}_{\frac{9\cdot 4-36-4}{4}=}=0$

$$(x+3)^2+(y-1)^2=1$$

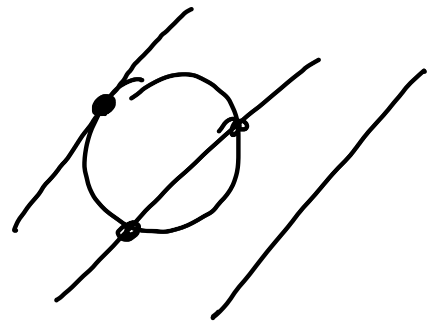
$$(x-(-3))^2+(y-1)^2=1$$

Center $C(-3,1)$, $r=1$

Intersection of a line and a circle

Find the intersection of the unit circle and the line $y = x + \frac{1}{2}$

$C(0,0) \quad r = 1$



$$x^2 + y^2 = 1$$

$$y = x + 1/2$$

$$x^2 + \left(x + \tfrac{1}{2}\right)^2 = 1$$

$$x^2 + x^2 + 2 \cdot x \cdot \tfrac{1}{2} + \tfrac{1}{4} - 1 = 0$$

$$\underbrace{2x^2}_{a} + \underbrace{1}_{b} \cdot x \underbrace{- 3/4}_{c} = 0$$

$$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 2\left(-\frac{3}{4}\right)}}{2 \cdot 2}$$

$$x = \frac{-1 \pm \sqrt{7}}{4}$$

$y = x + \frac{1}{2}$

If $x = \frac{-1+\sqrt{7}}{4}$ $y = \frac{-1+\sqrt{7}}{4} + \frac{1}{2}$ $P = \left(\frac{-1+\sqrt{7}}{4}, \frac{1}{4} + \frac{\sqrt{7}}{4}\right)$

if $x = \frac{-1-\sqrt{7}}{4}$, $y = \frac{-1-\sqrt{7}}{4} + \frac{1}{2}$ $Q\left(\frac{-1-\sqrt{7}}{4}, \frac{1}{4} - \frac{\sqrt{7}}{4}\right)$

In general line and circle can intersect in two points, one point, no points

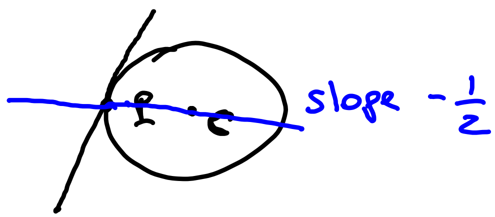
Tangent to a circle

Find the tangent to to the circle $(x-3)^2+(y+2)^2=\underset{}{\overset{r^2}{5}}$ at the point $P(1,-1)$

$(1-3)^2 + (-1+2)^2 = 5$?

$4 + 1 = 5$ ✓

P on circle.



Fact: Tangent line at P $\perp$ line PC

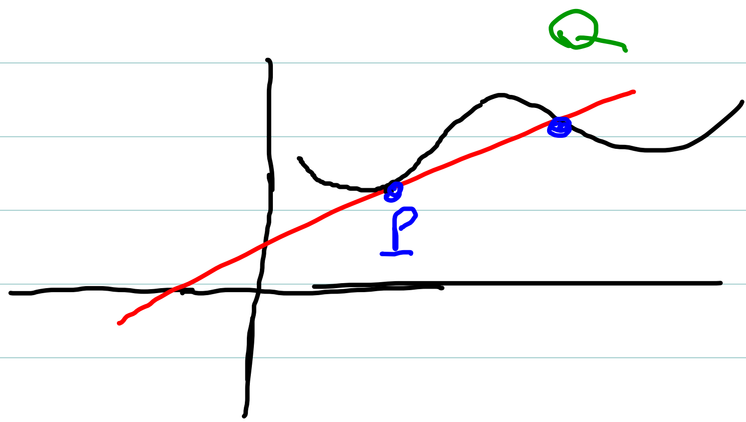
① Find slope of PC $(1,-1)\ (3,-2)$: $\dfrac{-1-(-2)}{1-3} = -\dfrac{1}{2}$

② Find slope of tangent line $m = -\dfrac{1}{\left(-\frac{1}{2}\right)} = 2$

③ Tangent line $\boxed{y = -1 + 2(x-1)}$

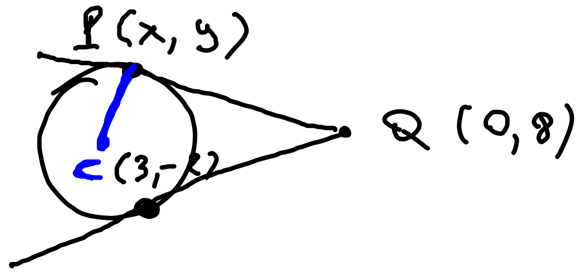
Note :

In general a curve and its tangent line at P may intersect at many points.



Find the tangent to to the circle $(x-3)^2+(y+2)^2=5$ through the point $Q(0,8)$

$(0-3)^2+(8+2)^2 = 109 > 5$



Find P

$\left\{\begin{array}{l} \text{Equation of circle} \\ \text{slope of tangent in way 1 } (\Delta y/\Delta x) = \text{slope of tangent way 2 } (\perp CP) \end{array}\right.$

$$\begin{cases} (x-3)^2+(y+2)^2=5 \\ \dfrac{y-8}{x-0} = -\dfrac{1}{\left(\frac{y+2}{x-3}\right)} \end{cases}$$

Slope of CP $\dfrac{y-(-2)}{x-3}$