Lesson 13

Read Chapter 10

Exponential functions

Given $g(x) = 2x - 1$ on the domain $0 \leq x \leq 5$. Is g invertible ? If it is find the inverse , its domain and its range

$y = 2x - 1$ solve for x

$$\frac{y+1}{2} = \frac{\cancel{2}x}{\cancel{2}}$$

$$g^{-1}(y) = \frac{y+1}{2}$$

or $x = 2y - 1$ solve for y

$$\frac{x+1}{2} = \frac{\cancel{2}y}{\cancel{2}}$$

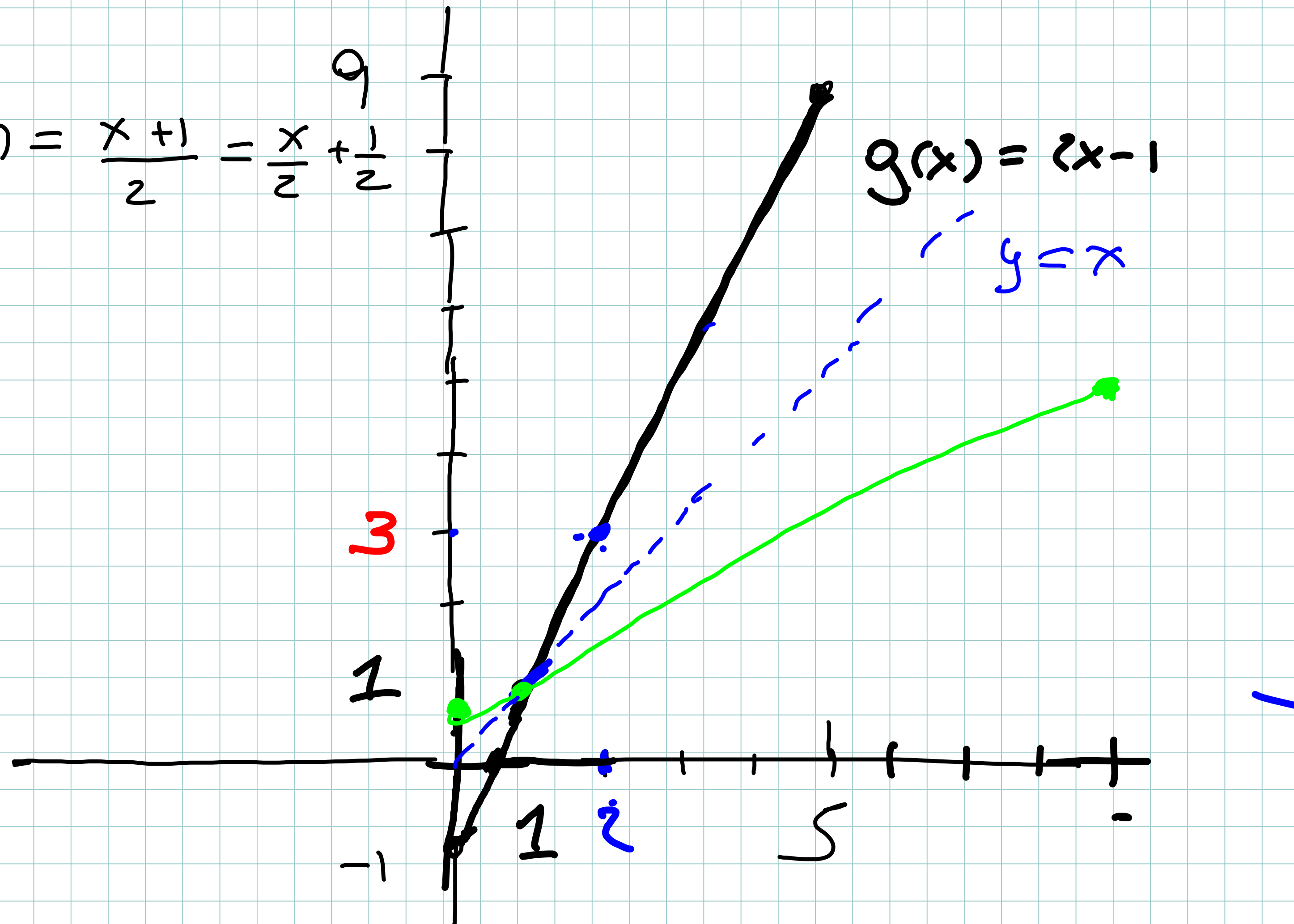
$$g^{-1}(x) = \frac{x+1}{2}$$

$$\begin{tikzcd} x \arrow[r, bend left, "g"] & y \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

Domain of g^{-1} = Range of g $[-1, 9]$

Range of g^{-1} = Domain of g $[0, 5]$

$$g^{-1}(x) = \frac{x+1}{2} = \frac{x}{2} + \frac{1}{2}$$



Suppose p is the price of an item and $q = f(p)$ is the number of items sold at that price. Explain in words the meaning of:

$$\begin{tikzcd} p \arrow[r, bend left, "f"] & q \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

prices # items sold

$f^{-1}(30) = 200 = p$ price

q # items sold

you sell 30 items, if each item has

price $p = \$200$

Function in standard exponential form : $f(x) = A_0\, a^x$, $a > 0$ and $a \neq 1$

$f(x) = 1000\,(-3)^x$ $\quad f(\tfrac{1}{2}) = 1000\,(-3)^{1/2} = 1000\sqrt{-3}$?

$1^x = 1$ $\quad f(x) = 1000 \cdot 1^x = 1000$

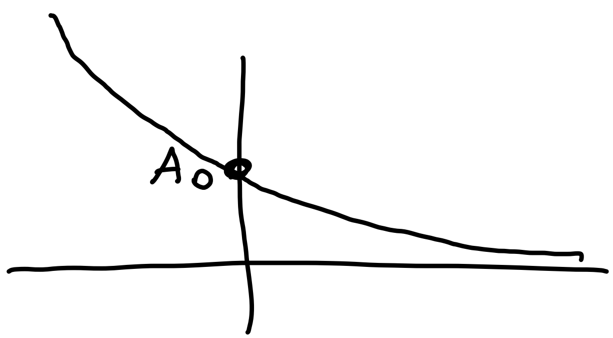
Example $f(x) = 1000 \cdot (1.3)^x$

$f(0) = A_0$

$$f(0) = A_0 \underbrace{a^0}_{1} = A_0$$

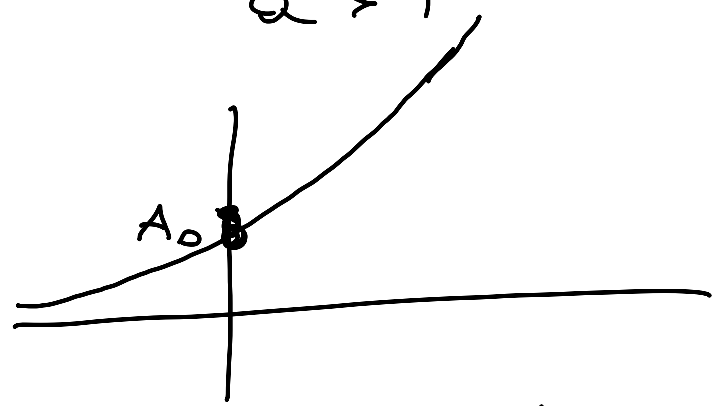
Graphs:

$0 < a < 1$



$f(x) = A_0 \left(\frac{1}{2}\right)^x$

$a > 1$



$f(x) = A_0\,(2)^x$

Useful algebra

1. $a^{x+y} = a^x \cdot a^y$ $\quad a^{x-y} = \frac{a^x}{a^y}$
2. $a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$
3. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
4. $a^{xy} = (a^x)^y$

Put $f(x) = 3 \cdot 2^{-x+\frac{1}{2}}$ in standard exponential form

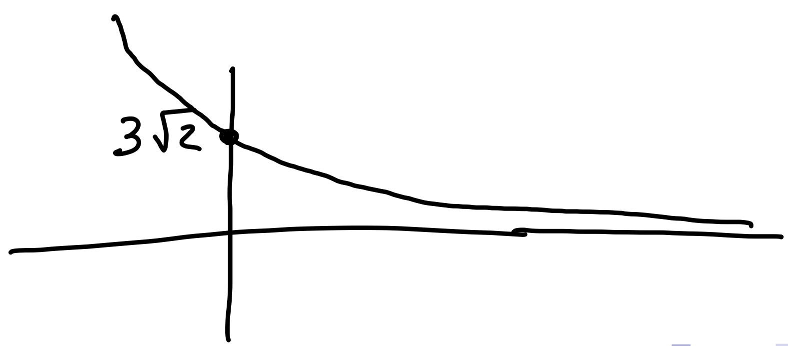
x^3 power

3^x exponential

$$f(x) = A_0 \, a^x$$

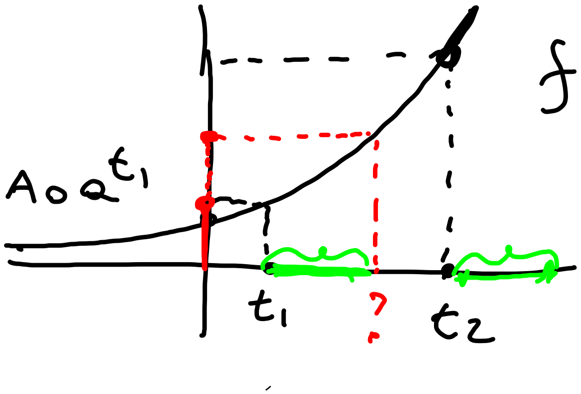
$$3 \cdot 2^{-x+\frac{1}{2}} = 3 \cdot 2^{-x} \cdot 2^{1/2} = 3\sqrt{2} \cdot 2^{-x}$$

$$\underbrace{3\sqrt{2}}_{A_0} \, {\underbrace{\left(\frac{1}{2}\right)}_{= a}}^{x}$$



Doubling time

Given an exponential function $f(t) = A_0\, a^{t}$, its doubling time is the period of time required for f to double in value.



$f(t) = A_0\, a^{t}$

Fact the doubling time does not depend on starting time

May as well look at $t=0$, $f(0) = A_0$

want to find time t s.t $f(t) = 2 \cdot A_0$

$$\cancel{A_0}\, a^{t} = 2 \cdot \cancel{A_0}$$

$$\ln(a^{t}) = \ln(2)$$

$$\frac{t \cdot \cancel{\ln(a)}}{\cancel{\ln(a)}} = \boxed{\frac{\ln(2)}{\ln(a)}}$$

Check that doubling time does not depend on initial time

starting time t_1 : $f(t_1) = A_0 a^{t_1}$

when is $f(t) = 2 A_0 a^{t_1}$?

$$A_0 a^t = 2 A_0 a^{t_1} .$$ solve for t

$$a^t = 2 a^{t_1}$$

$$\ln(a^t) = \ln(2 a^{t_1})$$

$$\ln(a^t) = \ln(2) + \ln a^{t_1}$$

$$t \ln a = \ln 2 + t_1 \ln a$$

$$t = \frac{\ln 2}{\ln a} + t_1$$

Note $\ln(2a^{t_1}) = t_1 \ln(2a)$ is WRONG ALGEBRA

The doubling time for $f(x) = A_0\, a^x$ is $\boxed{\frac{\ln 2}{\ln a}}$

$f(x) = A_0 \, a^x$ 2 points

$g(x) = mx + b$ 2 points

$h(x) = ax^2 + bx + c$ 3 points

Frequent questions:

1. Find an exponential function through two given points.
2. Find an exponential function through a given point, with a given doubling time.

Find a formula for the exponential function that passes through the points $(0, 2)$ and $(3, 5)$

A_0

$$f(x) = A_0 a^x$$

$$2 = A_0 a^0 \; ; \quad \boxed{2 = A_0}$$

$$5 = A_0 a^3 \; ; \quad 5 = 2 \cdot a^3 \; ; \quad \frac{5}{2} = a^3 \; ; \quad \boxed{\sqrt[3]{\frac{5}{2}} = a}$$

$$f(x) = 2 \left(\sqrt[3]{\frac{5}{2}} \right)^x$$

Find a formula for the exponential function that passes through the points $(1, 2)$ and $(3, 5)$

$$f(x) = A_0 a^x$$

$$2 = A_0 a$$

$$5 = A_0 a^3$$

$$\frac{5}{2} = \frac{\cancel{A_0} a^3}{\cancel{A_0} a} = a^2$$

$$\boxed{\sqrt{\frac{5}{2}} = a}$$

$$2 = A_0 \sqrt{\frac{5}{2}}$$

$$\boxed{\frac{2}{\frac{\sqrt{5}}{\sqrt{2}}} = A_0} = \frac{2\sqrt{2}}{\sqrt{5}}$$

$$f(x) = \frac{2\sqrt{2}}{\sqrt{5}} \left(\sqrt{\frac{5}{2}}\right)^x$$