Math 327 , Homework 2

1. Prove the following.

   (a) For any a, b real numbers, $|a| - |b| \leq |a + b|$.

   (b) For any a, b real numbers, $||a| - |b|| \leq |a + b|$.

   (c) For any a, b real numbers, $||a| - |b|| \leq |a - b|$.

   Note: (a) implies (b) and (b) implies (c) so if you do them in order, each will be a short proof.

2. Prove Bernoulli's Inequality
$$(1 + b)^n \geq 1 + nb$$
   in two different ways:

   (a) For any $b \geq 0$, using the binomial formula .

   (b) For any $b > -1$, using mathematical induction.

3. Decide if the following are true or false. If true, give a short proof. If false, find a counter example.

   (a) If the sequence $|a_n|$ converges, then so does (a_n).

   (b) If the sequence $(a_n + b_n)$ converges, then so do the sequences (a_n) and (b_n).

   (c) If the sequences $(a_n + b_n)$ and (a_n) converge, then so does the sequence (b_n).

4. Use the definition of convergence to show the following limits.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

   (b) $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2 + n} = 1.$

5. Discuss the convergence of the sequence $\left(\sqrt{n+1} - \sqrt{n}\right)_{n\in\mathbf{N}}$.

6. Let $a_1 = 1$ and for $n \geq 1$,
$$a_{n+1} = \begin{cases} \mathrm{a}_n + \frac{1}{n} & \text{if } \mathrm{a}_n^2 \leq 2 \\ \\ \mathrm{a}_n - \frac{1}{n} & \text{if } \mathrm{a}_n^2 > 2. \end{cases}$$
   Show that for every n, $|a_n - \sqrt{2}| < 2/n$ and prove that the sequence converges to $\sqrt{2}$.

7. For a sequence (a_n) of positive numbers, prove that
$$a_n \to \infty \text{ if and only if } \frac{1}{a_n} \to 0.$$
   Recall that and if and only if proof has two parts. You have to prove one side implies the other and vice versa.