Continuity and Uniform Continuity

Below I stands for any one of the intervals (a, b), $[a, b)$, $(a, b]$, $[a, b]$, (a, ∞), $[a, \infty)$, (∞, b), $(-\infty, b]$, $(-\infty, \infty) = \mathbf{R}$. Let f be a function defined on an interval I.

Definition of Continuity on an Interval: The function f is continuous on I if it is continuous at every c in I. So,
For every c in I, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - c| < \delta \qquad \text{implies} \qquad |f(x) - f(c)| < \epsilon.$$

If c is one of the endpoints of the interval, then we only check left or right continuity so $|x-c| < \delta$ is replaced by $0 < x - c < \delta$ or $0 < c - x < \delta$. Here, the δ may (and probably will) depend BOTH on c and ϵ.

Definition of Uniform continuity on an Interval The function f is uniformly continuous on I if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \qquad \text{implies} \qquad |f(x) - f(y)| < \epsilon.$$

Here, the δ may (and probably will) depend on ϵ but NOT on the points.

Uniform continiuty is stronger than continuity, that is,

Proposition 1 If f is uniformly continuous on an interval I, then it is continuous on I.

Proof: Assume f is uniformly continuous on an interval I.
To prove f is continuous at every point on I, let $c \in I$ be an arbitrary point.
Let $\epsilon > 0$ be arbitrary.
Let δ be the same number you get from the definition of uniform continuity.
Assume $|x - c| < \delta$.
Then, again from the definition of uniform continuity, $|f(x) - f(c)| < \epsilon$.
Therefore, f is continuous at c.
Since c was arbitrary, f is continuous everywhere on I.

The idea of the proof is basically that the δ you get for uniform continuity works for (regular) continuity at any point c, but not vice versa, since the δ you get for regular continuity may depend on the point c. When the interval is of the form $[a, b]$, uniform continuity and continuty are the same: f is continuous on $[a, b]$ if and only if f is uniformly continuous on $[a, b]$. This result is a combination of **Proposition 1** above with **Theorem B.4.4** in the book. I will leave you to read the proof of **Theorem B.4.4** on your own. It is optional. However, when the interval is not of the form $[a, b]$, the two are not necessarily the same. Below is an example of a function on an interval I which is continuous on I but not uniformly continuous on I.

Below are two proofs. For the first proof, write an explanation of why and how I wrote that particular line: Either as a necessary part of the proof (starting the proof, starting an implication proof, etc.) or explain how that particular line follows from a line above (which one?) using what algebra/theorem/rule. For the second proof, finish it by coming up with the contradiction. The contradiction must very VERY clear like $0 = 1$ or $1 < 1$. You wil do two similar ones as part of your homework.

Proposition 2: The function $f(x) = \dfrac{1}{x}$ is continous on the interval $(0, 1)$. That is, for every c in $(0, 1)$, f is continuous at c.

Proof: Let c be any number in $(0, 1)$.
Let $\epsilon > 0$ be given.
Define $\delta = \min\left\{\frac{c}{2}, \frac{c^2\epsilon}{2}\right\}$.
Assume $|x - c| < \delta$.
First, $-\dfrac{c}{2} < x - c < \dfrac{c}{2}$,
so $0 < \dfrac{c}{2} < x$
which implies $0 < \dfrac{1}{x} < \dfrac{2}{c}$.
Now,

$$\begin{aligned}
\left|\frac{1}{x} - \frac{1}{c}\right| &= \left|\frac{c - x}{cx}\right| \\
&= \frac{|x - c|}{cx} \\
&< \frac{1}{c} \cdot \frac{2}{c} \cdot |x - c| \\
&< \frac{2}{c^2} \cdot \frac{c^2\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$ and hence f is continuous at c.
Since $c \in (0, 1)$ was arbitrary,
f is continuous on $(0, 1)$.

Proposition 3: The function $f(x) = \dfrac{1}{x}$ is not uniformly continuous on the interval $(0, 1)$.

Proof:

For a contradiction, assume $f(x)$ is uniformly continuous on $(0, 1)$.
So, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \qquad \text{implies} \qquad |f(x) - f(y)| < \epsilon.$$

Now, let $\epsilon = \frac{1}{2}$. Then, there exists a $\delta > 0$ such that

$$|x - y| < \delta \qquad \text{implies} \qquad |f(x) - f(y)| < \frac{1}{2}.$$

Now, find a number $M > 1$ such that $\frac{1}{M} < \delta$. Let $x = \frac{1}{M}$ and $y = \frac{1}{M+1}$. Then,

$$|x - y| = \left|\frac{1}{M} - \frac{1}{M+1}\right| = \left|\frac{1}{M(M+1)}\right| < \frac{1}{M} < \delta$$

But,
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This is a contradiction. Therefore, f is not uniformly continuous on $(0, 1)$.