The following is a proof of Theorem 7.1.8 or Theorem B.3.2. It is essentially the same as the one in the book, with some minor variable name changes: I used y in place of t, c in place of a and e in place of b. The first one is because y = f(x) is standard, the last two are because I have my interval as (a, b) using up those letters. The main change is that I have more steps which are numbered so you can write the reasons referring back to previous steps, if necessary.

**Theorem 7.1.8** Let f be a one-to-one function differentiable on (a, b). Let \( c \in (a, b) \), \( f(c) = e \) and \( f'(c) \neq 0 \). Then, \( f^{-1} \) is differentiable at e and

\[
(f^{-1})'(e) = \frac{1}{f'(c)}.
\]

**Proof:** Let f be a one-to-one function differentiable on (a, b). Let \( c \in (a, b) \), \( f(c) = e \) and \( f'(c) \neq 0 \). To prove

\[
(f^{-1})'(e) = \frac{1}{f'(c)}
\]

we’ll show that

\[
\lim_{y \to e} \frac{f^{-1}(y) - f^{-1}(e)}{y - e} = \frac{1}{f'(c)}.
\]

Let \( \epsilon > 0 \) be given. First, the function \( g(z) = \frac{1}{z} \) is continuous at every \( z \neq 0 \), in particular at \( z = f'(c) \), so there exists a \( \delta_1 > 0 \) such that

\[
0 < |z - f'(c)| < \delta_1 \quad \text{implies} \quad \left| \frac{1}{z} - \frac{1}{f'(c)} \right| < \epsilon.
\]

Since f is differentiable at c,

\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)
\]

so using the definition of the limit with \( \epsilon = \delta_1 \), there is a \( \delta_2 > 0 \) such that

\[
0 < |x - c| < \delta_2 \quad \text{implies} \quad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \delta_1.
\]

Now, let \( x = f^{-1}(y) \) and \( c = f^{-1}(e) \). Since \( f^{-1} \) is continuous at e, using \( \epsilon = \delta_2 \) in the definition of continuity, there exists a \( \delta_3 > 0 \) such that

\[
0 < |y - e| < \delta_3 \quad \text{implies} \quad |f^{-1}(y) - f^{-1}(e)| < \delta_2.
\]

Now, let \( \delta = \delta_3 \).

For the rest of the steps, give a reason, completing computations if necessary:

1. Assume \( |y - e| < \delta \).

2. Then, \( |x - c| < \delta_2 \)

3. So

\[
\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \delta_1.
\]

4. Then,

\[
\left| \frac{1}{f(x) - f(c)} - \frac{1}{f'(c)} \right| < \epsilon.
\]

5. So,

\[
\left| \frac{f^{-1}(y) - f^{-1}(e)}{y - e} - \frac{1}{f'(c)} \right| < \epsilon.
\]
6. Therefore,

\[ 0 < |y - e| < \delta \]

implies

\[ \left| \frac{f^{-1}(y) - f^{-1}(e)}{y - e} - \frac{1}{f'(c)} \right| < \epsilon. \]

7. So

\[ \lim_{{y \to e}} \frac{f^{-1}(y) - f^{-1}(e)}{y - e} = \frac{1}{f'(c)}. \]

and

8.

\[ (f^{-1})' (e) = \frac{1}{f'(c)} \]