Taylor's Inequality for Taylor Polynomials

Taylor's Inequality states that the error, which is the difference between the actual value $f(x)$ and the approximate value $T_n(x)$ is bounded by

$$|f(x) - T_n(x)| \leq \frac{M|x-b|^{n+1}}{(n+1)!}$$

where $T_n(x)$ is the nth degree Taylor Polynomial approximating $f(x)$ near b and M is an upper bound for $|f^{(n+1)}(t)|$ for t between b and x.

1. Let $f(x) = e^x$ and $b = 0$. Then,

$$\boxed{f^{(n+1)}(t) = \qquad\qquad}$$

for **any** n. If x in in the interval $[-1, 1]$, and then, for any n we can take

$$\boxed{M = \qquad\qquad .}$$

(a) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = e^x$ using its

$$\boxed{T_4(x) = \qquad\qquad\qquad\qquad .}$$

is

$$\boxed{|e^x - T_4(x)| \leq \frac{M|x-b|^5}{5!} \leq \qquad\qquad .}$$

Compare this error bound with, for example, $|e^{0.9} - T_4(0.9)|$ on your calculator.

(b) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = e^x$ using its $T_n(x)$ is.

$$\boxed{|e^x - T_n(x)| \leq \frac{M|x-b|^{n+1}}{(n+1)!} \leq \qquad\qquad .}$$

Here,the answer will depend on n.

(c) If you want to approximate e^x with a Taylor polynomial so that $|e^x - T_n(x)| < 0.0005$, when x is in $[-2, 2]$, which n should you use? Hint: Using the answer to part (b) does not give not an equation you can solve for n (since it contains a factorial) , so you do this by trying $n = 5, n = 6...$ (you can see $n = 4$ does not work from part (a)) with the result of part (b) and stopping when you get the desired answer.

2. (This is a step by step version of a homework problem.) Let $f(x) = \ln(1-x)$ and $b = 0$. Then, take several derivatives

$f'(t) =$	$f''(t) =$	$f'''(t) =$
$f^{iv}(t) =$	$f^{v}(t) =$	$f^{vi}(t) =$

to get (guess) the formula for

$$f^{(n+1)}(t) =$$

Your answer should depend on n, should involve a factorial, and should match with your derivatives for the values of $n = 1, 2, 3, 4, 5$.

Now, let x be in the interval $[-0.5, 0.5]$. decide whether $|f^{(n+1)}(t)|$ is increasing of decreasing in the interval $[-0.5, 0.5]$ and then find an M which bounds $|f^{(n+1)}(t)|$ (possibly the maximum value)

$$|f^{(n+1)}(t)| \leq \qquad\qquad = M$$

Your answer will have n in it.

(a) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = \ln(1-x)$ using its $T_4(x)$ is

$$|\ln(1-x) - T_4(x)| \leq \frac{M|x-b|^5}{5!} \leq \qquad\qquad .$$

(b) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = \ln(1-x)$ using its $T_n(x)$ is

$$|\ln(1-x) - T_n(x)| \leq \frac{M|x-b|^{n+1}}{(n+1)!} \leq \qquad\qquad .$$

Here,the answer will depend on n.

(c) If you want to approximate $\ln(1-x)$ with a Taylor polynomial so that $|\ln(1-x) - T_n(x)| \leq 0.05$, when x is in $[-0.5, 0.5]$, which n should you use? Note that acceptable error in this part is different from the homework question.

3. Let $f(x) = \cos x$ and $b = 0$. Then

$$\boxed{\left|f^{(n+1)}(t)\right| = \qquad\qquad \text{or} \qquad\qquad}$$

so we can take

$$\boxed{M = \qquad}$$

regardless of the interval x belongs to.

(a) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = \cos x$ using its

$$\boxed{T_4(x) = \qquad\qquad\qquad\qquad .}$$

when x is in $[-0.5, 0.5]$ is

$$\boxed{|\cos x - T_4(x)| \leq \frac{M|x-b|^5}{5!} \leq \qquad\qquad\qquad .}$$

(b) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = \cos x$ using its

$$\boxed{T_5(x) = \qquad\qquad\qquad\qquad .}$$

when x is in $[-0.5, 0.5]$ is

$$\boxed{|\cos x - T_5(x)| \leq \frac{M|x-b|^6}{6!} \leq \qquad\qquad\qquad .}$$

(c) Using Taylor's Inequality, an upper bound for the error for approximating $f(x) = \cos x$ using its $T_n(x)$ is

$$\boxed{|\cos x - T_n(x)| \leq \frac{M|x-b|^{n+1}}{(n+1)!} \leq \qquad\qquad\qquad\qquad .}$$

Here,the answer will depend on n.

(d) If you want to approximate $\cos x$ with a Taylor polynomial so that $|\cos x - T_n(x)| \leq 10^{-7}$, when x is in $[-0.5, 0.5]$, which n should you use? Hint: Using the answer to part (b) does not give not an equation you can solve for n , so you do this by trying $n = 6, n = 7...$ with the result of part (b) and stopping when you get the desired answer.