## Taylor's Inequality for Quadratic Approximations

You have to complete the worksheet on Linear Approximations before doing this one. Problems below use ideas and results from there.

Taylor's Inequality states that the error, which is the difference between the actual value $f(x)$ and the approximate value $T_{2}(x)$ is bounded by

$$
\left|f(x)-T_{2}(x)\right| \leq \frac{M|x-b|^{3}}{3!}
$$

where $M$ is an upper bound for $\left|f^{\prime \prime \prime}(t)\right|$ for $t$ between $b$ and $x$.

1. Let $f(x)=e^{x}$ and $b=0$. Then $f^{\prime \prime \prime}(t)=e^{t}$ Since $f^{\prime \prime}=f^{\prime \prime \prime}$ are the same, all the $M$ computations will be the same as the corresponding question in the Linear Approximations worksheet because we are using the same numbers. The $|x-b|$ parts will also be the same.
(a) If $x=0.2$, then Taylor's inequality gives

$$
\left|e^{0.2}-T_{2}(0.2)\right| \leq \frac{M|x-b|^{3}}{6}=
$$

(b) If $x=-0.2$, then Taylor's inequality gives

$$
\left|e^{-0.2}-T_{2}(-0.2)\right| \leq \frac{M|x-b|^{3}}{6}=
$$

(c) If $x$ in in the interval $[-0.3,0.3]$, then Taylor's inequality gives

$$
\left|e^{x}-T_{2}(x)\right| \leq \frac{M|x-b|^{3}}{6} \leq
$$

(d) If $x$ in in the interval $[-a, a]$, where $a>0$, then Taylor's inequality gives

$$
\left|e^{x}-T_{2}(x)\right| \leq \frac{M|x-b|^{3}}{6} \leq
$$

(e) Compare the error bounds in (a)-(c) with the corresponding ones in Linear Approximations Question 1 parts (a)-(c). Do Linear or Quadratic Approximations have less error (bound), so more accuracy?
2. Let $f(x)=\ln (1+x)$ and $b=0$. Then $\left|f^{\prime \prime \prime}(t)\right|=\left|\frac{2}{(1+t)^{3}}\right|=\frac{2}{(1+t)^{3}}$, when $t>-1$. Now, $\frac{2}{(1+t)^{3}}$ is a decreasing function (WHY?), so it will have a maximum value for $t$ to the left of the interval. Note that the numbers below are different from the corresponding question in Linear Approximations!
(a) If $x=0.15$, then Taylor's inequality becomes

$$
\left|\ln (1+0.15)-T_{2}(0.15)\right| \leq \frac{M|x-b|^{3}}{6}=
$$

(b) Compute $T_{2}(x)$. Then, compare $\left|\ln (1+0.15)-T_{2}(0.15)\right|$ from your calculator with the bound you found above. Which one is more? Why?
(c) If $x$ in in the interval $[-0.25,0.25]$, then Taylor's inequality becomes

$$
\left|\ln (1+x)-T_{2}(x)\right| \leq \frac{M|x-b|^{3}}{6} \leq
$$

(d) If $x$ is in the interval $[-a, a]$, where $a>0$, then Taylor's inequality becomes

$$
\left|\ln (1+x)-T_{2}(x)\right| \leq \frac{M|x-b|^{3}}{6} \leq
$$

(e) Use part (d) above to find the largest interval $[-a, a]$ such that Taylor's Inequality guarantees that $\left|\ln (1+x)-T_{2}(x)\right| \leq 0.001$ for all $x$ in that interval.
3. Let $f(x)=\cos (x)$ and $b=0$. Then

$$
f^{\prime \prime \prime}(t)=
$$

so we can take

$$
M=
$$

regardless of the value of $x$ or the interval it belongs to.
(a) If $x$ in in the interval $[-a, a]$, where $a>0$, then Taylor's inequality will be

$$
\left|\cos (x)-T_{2}(x)\right| \leq \frac{M|x-b|^{3}}{6} \leq
$$

(b) Use part (b) above to find the largest interval $[-a, a]$ such that Taylor's Inequality guarantees that $\left|\cos x-T_{2}(x)\right| \leq 0.0005$ for all $x$ in that interval.

