Taylor's Inequality for Linear Approximations

Taylor's Inequality states that the error, which is the difference between the actual value $f(x)$ and the approximate value $T_1(x)$ is bounded by

$$|f(x) - T_1(x)| \leq \frac{M|x-b|^2}{2}$$

where M is an upper bound for $|f''(t)|$ for t between b and x.

1. Let $f(x) = e^x$ and $b = 0$. Then $f''(t) = e^t$ and $|e^t| = e^t$ because the exponential function is always positive. Now, e^t is an **increasing function**, so it will have a maximum value with t to the right of the interval.

   (a) If $x = 0.2$, then t will be in $[0, 0.2]$ so we can take M to be the maximum value of e^t on $[0, 0.2]$ which is

   $$\boxed{M = \qquad .}$$

   Using

   $$\boxed{|x-b| = \qquad ,}$$

   complete Taylor's inequality

   $$\boxed{\left|e^{0.2} - T_1(0.2)\right| \leq \frac{M|x-b|^2}{2} = \qquad .}$$

   (b) If $x = -0.2$, then t will be in $[-0.2, 0]$ so we can take M to be the maximum value of e^t on $[-0.2, 0]$ which is

   $$\boxed{M = \qquad .}$$

   Using

   $$\boxed{|x-b| = \qquad ,}$$

   complete Taylor's inequality

   $$\boxed{\left|e^{-0.2} - T_1(-0.2)\right| \leq \frac{M|x-b|^2}{2} = \qquad .}$$

   (c) If x in in the interval $[-0.3, 0.3]$, (This version is most common with x in an interval centered at b), then t could be anywhere in the **same** interval $[-0.3, 0.3]$ so we can take M to be the maximum value of e^t on $[-0.3, 0.3]$ which is

   $$\boxed{M = \qquad .}$$

   Using

   $$\boxed{|x-b| \leq \qquad ,}$$

complete Taylor's inequality

$$\boxed{|e^x - T_1(x)| \le \frac{M|x-b|^2}{2} \le \qquad\qquad .}$$

(d) If x in in the interval $[-a, a]$, where $a > 0$, then t could be anywhere in the **same** interval $[-a, a]$ so we can take M to be the maximum value of e^t on $[-a, a]$ which is (here M depends on a)

$$\boxed{M = \qquad .}$$

Using

$$\boxed{|x - b| \le \qquad ,}$$

(which should also depend on a) complete Taylor's inequality

$$\boxed{|e^x - T_1(x)| \le \frac{M|x-b|^2}{2} \le \qquad\qquad .}$$

2. Let $f(x) = \ln(1+x)$ and $b = 0$. Then $f''(t) = -\frac{1}{(1+t)^2}$ and $\left|-\frac{1}{(1+t)^2}\right| = \frac{1}{(1+t)^2}$ because $(1+t)^2$ is non-negative. Now, $\frac{1}{(1+t)^2}$ is a **decreasing function** (WHY?), so it will have a maximum value with t to the left of the interval.

(a) If $x = 0.2$, then t will be in $[0, 0.2]$ so we can take M to be the maximum value of $\frac{1}{(1+t)^2}$ on $[0, 0.2]$ which is

$$\boxed{M = \qquad .}$$

Using

$$\boxed{|x - b| = \qquad ,}$$

complete Taylor's inequality

$$\boxed{|\ln(1+0.2) - T_1(0.2)| \le \frac{M|x-b|^2}{2} = \qquad\qquad .}$$

(b) If $x = -0.2$, then t will be in $[-0.2, 0]$ so we can take M to be the maximum value of $\frac{1}{(1+t)^2}$ on $[-0.2, 0]$ which is

$$\boxed{M = \qquad .}$$

Using

$$\boxed{|x-b| = \qquad ,}$$

complete Taylor's inequality

$$\boxed{|\ln(1-0.2) - T_1(-0.2)| \leq \frac{M|x-b|^2}{2} = \qquad .}$$

(c) If x in in the interval $[-0.3, 0.3]$, then t could be anywhere in the **same** interval $[-0.3, 0.3]$ so we can take M to be the maximum value of $\dfrac{1}{(1+t)^2}$ on $[-0.3, 0.3]$ which is

$$\boxed{M = \qquad .}$$

Using

$$\boxed{|x-b| \leq \qquad ,}$$

complete Taylor's inequality

$$\boxed{|\ln(1+x) - T_1(x)| \leq \frac{M|x-b|^2}{2} \leq \qquad .}$$

(d) If x in in the interval $[-a, a]$, where $a > 0$. Then t could be anywhere in the **same** interval $[-a, a]$ so we can take M to be the maximum value of $\dfrac{1}{(1+t)^2}$ on $[-a, a]$ which is (here M depends on a)

$$\boxed{M = \qquad .}$$

Using

$$\boxed{|x-b| \leq \qquad ,}$$

(which should also depend on a) complete Taylor's inequality

$$\boxed{|\ln(1+x) - T_1(x)| \leq \frac{M|x-b|^2}{2} \leq \qquad .}$$

(e) Use part (d) above to find the largest interval $[-a, a]$ such that Taylor's Inequality guarantees that $|\ln(1+x) - T_1(x)| \leq 0.001$ for all x in that interval. Note that this is very similar to a question in the homework. What is different?

3. Let $f(x) = \sin(x)$ and $b = 0$. Then $f''(t) = -\sin(t)$ and $|-\sin(t)| = |\sin(t)|$. Here we get lazy and regardless of the interval simply use

$$\boxed{M = 1}$$

since $|\sin t| \leq 1$ for any t.

(a) If $x = 0.2$, then

$$\boxed{|x - b| = \qquad\quad ,}$$

so Taylor's inequality will be

$$\boxed{|\sin(0.2) - T_1(0.2)| \leq \frac{M|x-b|^2}{2} = \qquad\qquad .}$$

(b) If x in in the interval $[-0.3, 0.3]$. Then,

$$\boxed{|x - b| \leq \qquad\quad ,}$$

so Taylor's inequality will be

$$\boxed{|\sin(x) - T_1(x)| \leq \frac{M|x-b|^2}{2} \leq \qquad\qquad .}$$

(c) If x in in the interval $[-a, a]$, where $a > 0$, then

$$\boxed{|x - b| \leq \qquad\quad ,}$$

(which depends on a) so Taylor's inequality will be

$$\boxed{|\sin(x) - T_1(x)| \leq \frac{M|x-b|^2}{2} \leq \qquad\qquad .}$$

(d) Use part (d) above to find the largest interval $[-a, a]$ such that Taylor's Inequality guarantees that $|\sin x - x| \leq 0.001$ for all x in that interval.