$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad (\text{mostly } b=0)$$

definition

Taylor Series Answers:

$$f(x) = \sum \underbrace{\sim\!\sim\!\sim}_{\text{coefficients}} (x-b)^{\sim}$$

ex 1 Find the Taylor Series for

$$f(x) = \frac{e^x - 1}{x} \quad \text{near } b=0.$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^x - 1 = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) - 1 = \left(1 + x + \frac{x^2}{2!} + \cdots\right) - 1$$

$$= \sum_{k=1}^{\infty} \frac{x^k}{k!} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1}{x} = \frac{1}{x}\sum_{k=1}^{\infty} \frac{x^k}{k!} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}{x}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} x^{k-1} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

— o —

you can do a change of variable

let $n = k-1$

when $k=1$
$n = k-1 = 1-1 = 0$

$n+1 = k$

$$\frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n$$

— o —

ex ② same question, $f(x) = \frac{e^x + e^{-x}}{2}$

$$f(x) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k + \sum_{k=0}^{\infty} \frac{1}{k!} (-x)^k\right)$$

$$f(x) = \frac{1}{2}\left(\left(1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} + \cdots\right) + \left(1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} - \cdots\right)\right)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k\right)$$

$$f(x) = \frac{1}{\cancel{2}}\left(\cancel{2}\cdot 1 + \cancel{2}\cdot\frac{x^2}{2!} + \cancel{2}\cdot\frac{x^4}{4!} + \cancel{2}\cdot\frac{x^6}{6!} + \cdots\right)$$

$$\boxed{f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}$$

Looking ↗ I construct Σ version:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$$

(x anything)

—o—

side note: $f(x) = \cosh(x)$
hyperbolic cosine

hyperbolic sine
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

ex: $f(x) = \dfrac{1}{5+2x} - \dfrac{1}{3-7x}$

$$= \frac{1}{5}\left(\frac{1}{1+\frac{2}{5}x}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{7}{3}x}\right)$$

$$= \frac{1}{5}\left(\frac{1}{1-\left(\frac{-2}{5}x\right)}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{7}{3}x}\right)$$

$\left|\frac{-2}{5}x\right|<1$ and $\left|\frac{7}{3}x\right|<1$

GS

$$= \frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{-2}{5}x\right)^k - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{7}{3}x\right)^k$$

same

$|x|<\frac{5}{2}$ and $|x|<\frac{3}{7}$

clean

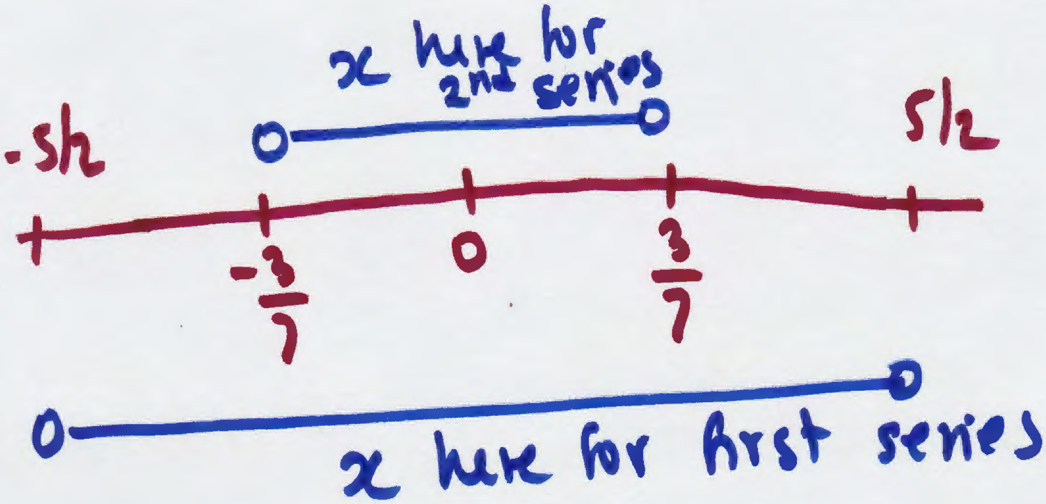
$$= \sum_{k=0}^{\infty}\frac{(-2)^k}{5^{k+1}}x^{\underline{\underline{k}}} - \sum_{k=0}^{\infty}\frac{7^k}{3^{k+1}}x^{\underline{\underline{k}}}$$

same

$$= \sum_{k=0}^{\infty}\left(\frac{(-2)^k}{5^{k+1}} - \frac{7^k}{3^{k+1}}\right)x^k$$

$$f(x) = \underbrace{\left(\frac{1}{5} - \frac{1}{3}\right)}_{k=0} + \underbrace{\left(\frac{-2}{25} - \frac{7}{9}\right)x}_{k=1} + \underbrace{\left(\frac{4}{125} - \frac{49}{27}\right)x^2}_{k=2} + \dots$$

$$= \frac{-2}{15} - \frac{193}{225}x - \frac{6017}{3375}x^2 + \dots$$



I want **both** to hold.

So: $|x| < \frac{3}{7}$

or $-\frac{3}{7} < x < \frac{3}{7}$.

ex: $f(x) = \dfrac{x}{(1+x^2)^2}$

First idea: $x \cdot \underbrace{\dfrac{1}{1+x^2} \cdot \dfrac{1}{1+x^2}}$

I don't like multiplying series.

After staring at $f(x)$ for a while you realize it's the derivative of something

$$f(x) = \frac{d}{dx}\left(\frac{-1}{2}\,\frac{1}{1+x^2}\right)$$

$$= \frac{-1}{2}\,\frac{d}{dx}\left(\frac{1}{1-(-x^2)}\right)$$

$|-x^2|<1$ $$\overset{GS}{=} \frac{-1}{2}\,\frac{d}{dx}\left(\sum_{k=0}^{\infty} (-x^2)^k\right)$$

$$f(x) = \frac{-1}{2}\,\frac{d}{dx}\left(1 - x^2 + x^4 - x^6 + x^8 - \dots\right)$$

$$f(x) = -\frac{1}{2}\left(0 - 2x + 4x^3 - 6x^5 + 8x^7 - \dots\right)$$

$$f(x) = -\frac{1}{2}\sum_{k=0}^{\infty} \frac{d}{dx}(-1)^k x^{2k}$$

$$= -\frac{1}{\cancel{2}}\sum_{k=0}^{\infty} (-1)^k \cancel{2}k\, x^{2k-1}$$

when $k=0$
term $=0$, disappears

$$= \sum_{k=0}^{\infty} (-1)^{k+1} k\, x^{2k-1} = \sum_{k=1}^{\infty} (-1)^{k+1} k\, x^{2k-1}$$

$$f(x) = -\left(-x + 2x^3 - 3x^5 + 4x^7 - \dots\right)$$

$$= x - 2x^3 + 3x^5 - 4x^7 + \dots$$

$|x| < 1$.

$$\sum_{n=0}^{\infty} (-1)^n (n+1)\, x^{2n+1}$$

ex: $f(x) = \arctan(x) = \tan^{-1} x$

I can't see how to relate this to my 4 series so I go old fashioned:

$f(x) = \tan^{-1} x \qquad f(0) = 0$

$f'(x) = \boxed{\dfrac{1}{1+x^2}}$ GS! $\qquad f'(0) =$

$f''(x) \qquad f''(0) =$

pattern?

IDEA: $\arctan x = \displaystyle\int \frac{1}{1+x^2}\,dx$

$|x^2| < 1$ (see previous problem) $\overset{GS}{=} \displaystyle\int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx$

$\arctan x = \displaystyle\int 1 - x^2 + x^4 - x^6 + x^8 - \dots\,dx$

$= x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots + C$

$C = f(0)$ (in general, $C = f(b)$)

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\, dx + C$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C$$

$$C = \arctan 0 = 0.$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\arctan x \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad |x| < 1.$$

qushon: $f(x) = \arctan(x)$

$$f^{(1999)}(0) = ?$$