Last Lecture - Finish Taylor Series

$$f(x) \underset{\substack{\uparrow \\ \text{definition}}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(b)}{k!} (x-b)^k$$

(most of the time $b=0$)

Getting $f^{(k)}(b)$ might be not practical.

We can manipulate known series for $\sin x, \cos x, e^x, \frac{1}{1-x}$ to get new ones.

We can ✓ - add/subtract
✓ - substitute for x
✓ - differentiate/integration

avoid - multiply ☹ divide ☹

ex: Find the series for $f(x) = \sin x \cos x$ about $b=0$.

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\right) \overset{?}{=}$$

$\frac{1}{2} + \frac{1}{6}$

$$\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= x + \left(-\frac{x^3}{2!} - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} + \frac{x^5}{2!3!} + \frac{x^5}{4!}\right) +$$

$= x + \left(-\frac{2}{3}\right)x^3 + \left(\frac{1}{120} + \frac{1}{12} + \frac{1}{24}\right)x^5 + \dots$

multiplication is not practical.

Alternatively, $f(x) = \sin x \cos x = \frac{1}{2}\sin(2x)$

$$\frac{1}{2}\sin(2x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k+1}}{(2k+1)!} \qquad \frac{2^{2k+1}}{2}$$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k 2^{2k}}{(2k+1)!}} x^{2k+1} = x - \frac{4}{6}x^3 + \dots \text{ all } x$$

Answers to Taylor Questions should always look like

$$\sum \left(\quad \right) \quad x^{\sim}$$

coefficients $\quad$ powers of x

ex: $f(x) = \dfrac{e^x + e^{-x}}{2}$

same

$$f(x) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \right) = \frac{1}{2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)+\left(1-x+\frac{x^2}{2!}\dots\right.\right.$$

same

odd terms will have opposite sign

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\left(1+(-1)^k\right)}{k!} x^k \qquad = \frac{1}{2}\left[2 + 2\left(\frac{x^2}{2!}\right) + 2\left(\frac{x^4}{4!}\right) \dots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

~~even~~ odd terms dissappear

even terms will double

$2 \wedge \frac{k}{2}$

$$= \sum_{i\,\cancel{k}=0}^{\infty} \frac{1}{(2i)!} x^{2i} \qquad \text{for } \underline{\text{all}}\ x$$

extra information: $f(x) = \cosh(x)$
hyperbolic cosine

we also have $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

ex: $f(x) = \dfrac{1}{5+2x} - \dfrac{1}{3-7x}$

$\dfrac{1}{1-A}$ $\quad = \dfrac{1}{5}\left(\dfrac{1}{1-\left(-\frac{2}{5}x\right)}\right) - \dfrac{1}{3}\left(\dfrac{1}{1-\frac{7}{3}x}\right)$

$\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$, $|x|<1$ Geometric Series

$-1 \le x \le 1$

$$= \frac{1}{5}\sum_{k=0}^{\infty}\left(-\frac{2}{5}x\right)^k - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{7}{3}x\right)^k$$

$\left|-\frac{2}{5}x\right| < 1$ and $\left|\frac{7}{3}x\right| < 1$

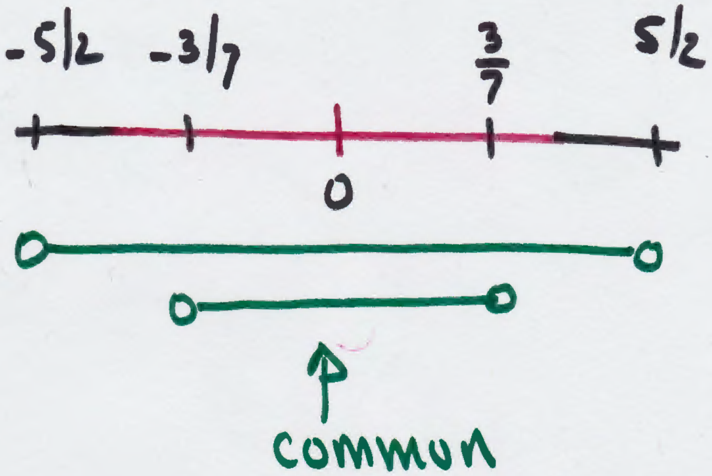
$$= \sum_{k=0}^{\infty}\frac{(-2)^k}{5^{k+1}}\,x^k - \sum_{k=0}^{\infty}\frac{7^k}{3^{k+1}}\,x^k$$

$|x| < \frac{5}{2}$ and $|x| < \frac{3}{7}$

$$\cancel{= \sum_{k=0}^{\infty}\left[\frac{(-2)^k}{5^{k+1}} - 7^k\right] x^k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{(-2)^k}{5^{k+1}} - \frac{7^k}{3^{k+1}} \right) x^k \quad , \quad |x| < \frac{3}{7}.$$

note $\frac{3}{7} < \frac{5}{2}$



$-\frac{3}{7} < x < \frac{3}{7}$

i.e. the radius of convergence is $\frac{3}{7}$.

If I need several terms:

k=0 k=1 k=2

$$T_2(x) = \left(\frac{1}{5} - \frac{1}{3}\right) + \left(\frac{-2}{25} - \frac{7}{9}\right)x + \left(\frac{4}{125} - \frac{49}{27}\right)x^2$$

ex: $f(x) = \arctan(x) = \tan^{-1}(x)$, $b=0$

I can't use $e^x, \sin x, \cos x, \frac{1}{1-x}$

$f(x) = \arctan(x)$ $\quad$ $f(0) = 0$

$\boxed{f'(x) = \frac{1}{1+x^2}}$ $\quad$ $f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\quad$ $f''(0) = 0$

f''', f^{iv} will be ugly.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k \quad , |-x^2|<1$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k} \quad , |x|<1.$$

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt \quad \text{OR} \quad \arctan(x) = \int \frac{1}{1+x^2}\,dx$$

$$\int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx \underset{\text{math magic}}{=} \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\,dx$$

$|x|<1$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C$$

integration constant

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C = \arctan(x)$$

when $x = 0$ this will be $0 \Rightarrow C = 0$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \quad , \ |x| < 1$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

ex: $f(x) = \dfrac{x}{(1+x^2)^2}$, $b=0$

one way: $x \cdot \underbrace{\dfrac{1}{1+x^2} \cdot \dfrac{1}{1+x^2}}_{\text{multiplying series}}$

☹

I realize $f(x)$ looks like $\dfrac{d}{dx} \dfrac{1}{1+x^2}$

$$f(x) = \frac{-1}{2} \frac{d}{dx} \frac{1}{1+x^2}$$

$$= -\frac{1}{2} \frac{d}{dx} \sum_{k=0}^{\infty} (-1)^k x^{2k} \quad , |x|<1$$

$$= -\frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dx} x^{2k} \quad , |x|<1$$

$$= -\frac{1}{2} \sum_{k=0}^{\infty} (-1)^k (2k) \, x^{2k-1} \quad , |x|<1$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} k \; x^{2k-1} = x - 2x^3 + 3x^5 - \dots$$

ex: $f(x) = \dfrac{1}{(1+x)^2} = -\dfrac{d}{dx}\dfrac{1}{1+x}$

last question/example: If $f(x) = \arctan(3x)$.

Find $f^{(2010)}(0) = ?$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad |x|<1$$

$$\arctan(3x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (3x)^{2k+1}, \quad |3x|<1$$

$$\boxed{= \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{2k+1} x^{2k+1} = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i}, \quad |x| < \frac{1}{3}$$

Same so all terms must match!

$$\cdots + \frac{f^{(2010)}(0)}{2010!} x^{2010} + \cdots$$

2k+1 is always odd. 2010 is even

so $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{2k+1} x^{2k+1}$ has <u>no</u> even

terms. so the coefficient of x^{2010}

is 0.

$$\rightarrow \frac{f^{(2010)}(0)}{2010!} = 0$$

$$\rightarrow f^{(2010)}(0) = 0.$$

Let's try $f^{(2011)}(0)$?

$2k+1 = 2011 \rightarrow 2k = 2010 \rightarrow k = 1005$

$$\frac{(-1)^{1005} 3^{2011}}{2011} x^{2011} = \frac{\boxed{f^{(2011)}(0)}}{2011!} x^{2011}$$

$$\frac{(-3^{2011})\, 2011!}{2011} = f^{(2011)}(0)$$

$$\frac{n!}{n} = \frac{\cancel{n}\cdot(n-1)\cdot(n-2)\cdot\ldots 3\cdot 2\cdot 1}{\cancel{n}} = (n-1)!\ , n \geq 1$$

$$-3^{2011}\cdot 2010! = f^{(2011)}(0)$$

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Stay safe + stay healthy!

E. Belyea