$$f(x) \approx T_n(x)$$

ex: $$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = T_3(x).$$

approximation improves as n gets larger

IDEA: $$f(x) \stackrel{?}{=} \lim_{n\to\infty} T_n(x)$$

Yes, most of the time, maybe with some restriction.

Section 4 - Taylor Series.

$$T_n(x) = f(b) + f'(b)(x-b) + \cdots + \frac{f^{(n)}(b)(x-b)^n}{n!}$$

$$= \sum_{i=0}^{n} \frac{f^{(i)}(b)(x-b)^i}{i!}$$

Taylor Polynomial.

Taylor Series is its limit as $n \to \infty$.

$$T(x) = f(b) + f'(b)(x-b) + \cdots$$ never stop

$$= \lim_{n \to \infty} T_n(x)$$

$$= \sum_{i=0}^{\infty} \frac{f^{(i)}(b)}{i!} (x-b)^i$$

Next, we'll show for $f(x) = e^x$ ($b=0$) the limit exists and equals e^x as claimed.

We'll show $\lim_{n \to \infty} T_n(x) = e^x$

by showing $\lim_{n \to \infty} | e^x - T_n(x) | = 0$.

Use Sandwich | Squeeze Theorem

$$0 \le |e^x - T_n(x)| \le \frac{M|x|^{n+1}}{(n+1)!} \to 0$$

0

error upper bound

For M:

$$f^{(n+1)}(t) = e^t$$

$$|e^t| \le e^{|x|}$$

t is between 0 + x

we have $x \le |x|$

so $t \le |x|$

no n here

$$0 \le |e^x - T_n(x)| \le \frac{\boxed{e^{|x|}}|x|^{n+1}}{(n+1)!}$$

claim: $\xrightarrow{n\to\infty} 0$

I will be done if I can show

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \quad \text{for any } x.$$

~~ex~~ persuation by example:

let $|x| = 5$

$n=1$ $\qquad$ $n=2$ $\qquad$ $n=3$

$$\frac{5^2}{2} \qquad \frac{5^3}{6} \qquad \frac{5^4}{24} \quad \cdots$$

$\cdots\cdots$ $n=10$

$$\frac{5\,5\,5\,5\cdot5\,5\,5\cdot5\cdot5\cdot5\cdot5}{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\,4\,3\,2\,1}$$

$\cdots\cdots$ $n=1000$

$$\frac{5\cdot5\cdot5\cdot5 \quad \cdots 5\,5}{1001\cdot1000\cdot999\cdot998} \longrightarrow 0.$$

You can easily modify this argument for $\sin x$ + $\cos x$ with $M=1$ and show

$\lim_{n\to\infty} T_n(x) = \sin x$, $\lim_{n\to\infty} T_n(x) = \cos x$.

Why do we need these series?

ex: Find $T_7(x)$ for $f(x) = \frac{1}{1+x^2}$

near $b=0$.

$$f(x) = \frac{1}{1+x^2}$$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

$$f''(x) = \frac{-2(1+x^2)^2 - (-2x)2(1+x^2)2x}{(1+x^4)^4}$$

 $f'''(x) =$

I don't want to do this.

Four Common Series (you need)

① $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad \text{for all } x.$$

② $$\sin x = \overbrace{x - \frac{x^3}{3!}}^{T_3(x) = T_4(x)} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \quad \text{for all } x$$

③ $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} \quad \text{For all } x$$

④ $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

(limit exists only)

$\boxed{|x| < 1.}$

i.e. $-1 < x < 1.$

We'll take these 4 series & we'll do anything that is mathematically correct

- addition/subtraction
- multiplication 😐
- substitution
- differentiation / integration

to get new ones.

ex: Find the Taylor Series for

$f(x) = \frac{1}{1-5x}$ near $b=0$.

Then write down $T_4(x)$ for this function.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad |x|<1$$

so $$\frac{1}{1-(5x)} = 1 + 5x + (5x)^2 + (5x)^3 + (5x)^4 + \dots \quad |5x|<1$$

clean up $$\boxed{\frac{1}{1-5x} = 1 + 5x + 5^2x^2 + 5^3x^3 + 5^4x^4 + \dots \quad |x|<\frac{1}{5}}$$

$\frac{1}{5}$ is called the radius of convergence

Then,

$$-\frac{1}{5} < x < \frac{1}{5}$$

this is called the interval of convergence.

Solution in Σ notation:

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i, \quad |x|<1$$

$$\frac{1}{1-5x} = \sum_{i=0}^{\infty} (5x)^i = \boxed{\sum_{i=0}^{\infty} 5^i x^i, \quad |x|<\frac{1}{5}.}$$

ex: $f(x) = \frac{1}{1+x^2}$, $b=0$

Find Taylor Series and $T_7(x)$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots$$

$|-x^2| < 1$

$$\boxed{\frac{1}{1+x^2} \underset{\text{clean up}}{=} 1 - x^2 + x^4 - x^6 + \dots, \quad |x| < 1.}$$

Solution in Σ notation

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{i=0}^{\infty} (-x^2)^i, \quad |-x^2| < 1$$

$$\frac{1}{1+x^2} = \sum_{i=0}^{\infty} (-1)^i x^{2i}, \quad |x| < 1$$

radius of convergence

Then

$$T_7(x) = 1 - x^2 + x^4 - x^6$$

(I stopped before x^8)

highest power of $(x-b)$ must be 7

ex: $f(x) = \dfrac{2}{5+4x}$, $b=0$

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i, \quad |x|<1$$

$$\frac{2}{5+4x} = 2\left(\frac{1}{5+4x}\right) = \frac{2}{5}\left(\frac{1}{1+\frac{4}{5}x}\right)$$

$$= \frac{2}{5}\left(\frac{1}{1-\left(-\frac{4}{5}x\right)}\right) = \frac{2}{5}\sum_{i=0}^{\infty}\left(-\frac{4}{5}x\right)^i, \quad \left|-\frac{4}{5}x\right|<1$$

$$= \sum_{i=0}^{\infty}(-1)^i\frac{2\cdot 4^i}{5^{i+1}}x^i \underset{\text{optional}}{=} \sum_{i=0}^{\infty}\frac{(-1)^i 2^{2i+1}}{5^{i+1}}x^i$$

answer

$|x| < \frac{5}{4}$