TN3 – T_3 + higher order Taylor Polynomials.

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \int_b^x f'''(t)\frac{(x-t)^2}{2}\,dt$$

Do By Parts AGAIN $\quad u = f'''(t) \quad dv = \frac{(x-t)^2}{2}dt$

$$f(x) = \underbrace{f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f'''(b)}{6}(x-b)^3}_{T_3(x)} + \int_b^x f^{IV}(t)\frac{(x-t)^3}{6}\,dt$$

$$|f(x) - T_3(x)| \leq \frac{M|x-b|^4}{24}$$

$|f^{IV}(t)| \leq M$ where t is between b + x

Integration by parts AGAIN

$$f(x) = \frac{f(b)}{0!} + \frac{f'(b)}{1!}(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f'''(b)}{6}(x-b)^3$$

$$+ \frac{f^{iv}(b)}{24}(x-b)^4 + \int_b^x f^{v}(t)\frac{(x-t)^4}{24}\,dt$$

Notation

(1) For derivatives

$f', f'', f''', f^{iv}, f^{v}, \ldots$ ← Roman Numerals

better

$f = f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}, \ldots\ldots, f^{(n)}, f^{(n+1)},$

(2) Denominators $1, 2, 6 = 2\cdot3, 24 = 2\cdot3\cdot4,$

$1\cdot2\cdot3\cdot\ldots\cdot n = n!$ "n factorial"

n non-negative integer.

$0! = 1$

(3) Summation (Sigma) Notation

$$1+2+3+\dots+100 = \sum_{i=1}^{100} i$$

$$1+4+9+16+\dots+100 = \sum_{i=1}^{10} i^2$$

$$\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2048} = \sum_{i=2}^{11} \frac{1}{2^i}$$

$$1+x+x^2+x^3+\dots+x^n = \sum_{i=0}^{n} x^i$$

$$T_n(x) = f(b)+f'(b)(x-b)+\frac{f''(b)}{2}(x-b)^2$$

$$+\dots+\frac{f^{(n)}(b)}{n!}(x-b)^n$$

$$=\sum_{i=0}^{n} \frac{f^{(i)}(b)}{i!}(x-b)^i$$

The Error Upper Bound
(Taylor's Inequality)

$$|f(x) - T_n(x)| \leq \frac{M\,|x-b|^{n+1}}{(n+1)!}$$

$$|f^{(n+1)}(t)| \leq M$$ where t is between b & x

example: 1) Find $T_n(x)$ for $f(x) = e^x$ near $b = \underline{\underline{0}}$.

2) Estimate $\sqrt{e}$ using $T_5(x)$.

3) Find an upper bound for the error in approximating $e^x \approx T_5(x)$ if x is in $[-\frac{1}{2}, \frac{1}{2}]$.

1) $f(x) = e^x \quad f'(x) = e^x \quad f''(x) = e^x \quad \ldots \quad f^{(n)}(x) = e^x$

$f(0) = 1 \quad f'(0) = 1 \quad f''(0) = 1 \quad \ldots \quad f^{(n)}(0) = 1$

$$T_n(x) = 1 + 1\cdot(x-0) + \frac{1}{2}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 + \cdots + \frac{1}{n!}(x-0)^n$$

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!}$$

2) $T_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$

$$\sqrt{e} = e^{1/2} = f\left(\tfrac{1}{2}\right) \approx T_5\left(\tfrac{1}{2}\right) = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{16\cdot 24} + \frac{1}{32\cdot 120} \underset{\text{calc.}}{\approx}$$

3) $|e^x - T_5(x)| \le \frac{M|x-0|^6}{6!} \le \frac{M\left(\frac{1}{2}\right)^6}{6!}$

$|f^{(6)}(t)| = e^t \underset{-\frac{1}{2} \le t \le \frac{1}{2}}{\le} e^{1/2}$ because e^t is increasing.

M

$$|e^x - T_5(x)| \le \frac{e^{1/2}}{64\cdot 720} \underset{\text{calc.}}{\approx}$$